

p. 52

UFam( $\omega$ -Sets) ( $\neq$  category)

$$\text{id}_{(X_i, E_i)_{i \in I, E}} = (\text{id}_{(I, E)}, \underbrace{\{\text{id}_{(X_i, E_i)}\}_{i \in I}}_{\text{uniformly}})$$

$\{\text{id}_{(X_i, E_i)}\}_{i \in I}$  is tracked by  $\Delta n, \Delta m, m$

$$(X_i, E_i)_{i \in (I, E)} \xrightarrow{(u, \{f_i\}_{i \in I})} (Y_j, E_j)_{j \in (J, E)} \xrightarrow{(v, \{g_j\}_{j \in J})} (Z_k, E_k)_{k \in (K, E)}$$

$$\underbrace{\hspace{15em}}_{(v \circ u, \{g_{u(i)} \circ f_i\})}$$

( $u$  and  $v$  are tracked by  $e_u$  and  $e_v$   
 $\{f_i\}_{i \in I}$  and  $\{g_j\}_{j \in J}$  are tracked by  $e_f$  and  $e_g$   
uniformly)

$\Rightarrow v \circ u$  is tracked by  $\Delta n, e_v \cdot (e_u \cdot m)$   
 $\{g_{u(i)} \circ f_i\}_{i \in I}$  is tracked uniformly by  
 $e = \Delta n, \Delta m, (e_g \cdot (e_u \cdot n)) \cdot ((e_f \cdot n) \cdot m)$

$$\forall i \in I, \forall n \in E(i), \forall x \in X_i, \forall m \in E(x).$$

$$e \cdot n \cdot m \in E_{v(u(i))} (g_{u(i)}(f_i(x)))$$

が成り立つのは明らか

p. 54

$$\mathbb{E}(X, Y) \cong \coprod_{u: pX \rightarrow pY} \mathbb{E}_{pX}(X, u^*Y)$$

of naturality

$$\begin{array}{ccc} f \in \mathbb{E}(X, Y) & \cong & \coprod_{u: pX \rightarrow pY} \mathbb{E}_{pX}(X, u^*Y) \ni (pf, f') \\ \mathbb{E}(g', h) & & \downarrow \\ \downarrow & & \\ h \circ f \circ g \in \mathbb{E}(X', Y') & \cong & \coprod_{v: pX' \rightarrow pY'} \mathbb{E}_{pX'}(X', v^*Y') \end{array}$$

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ & & \downarrow f' \\ u^*Y & \xrightarrow{\bar{u}} & Y \xrightarrow{h} Y' \end{array}$$

$$pX' \xrightarrow{pg} pX \xrightarrow{u} pY \xrightarrow{ph} pY'$$

# Exercise 1.4.1

$$\begin{array}{ccc}
 \mathbb{E} & \pi^*(X) \xrightarrow{\bar{\pi}} X & \delta^*(X) \xrightarrow{\bar{\delta}} X \\
 p \downarrow & & \\
 \mathbb{B} & \bar{I} \times \bar{J} \xrightarrow{\pi} \bar{I} & \bar{I} \times \bar{J} \xrightarrow[\delta = \langle id, \pi' \rangle]{} (\bar{I} \times \bar{J}) \times \bar{J}
 \end{array}$$

Family fibration

$$\begin{aligned}
 \pi^*(\{X_i\}_{i \in \bar{I}}) &= \{X_{(i,j)}\}_{(i,j) \in \bar{I} \times \bar{J}} \\
 \bar{\pi}(\{X_i\}_{i \in \bar{I}}) &= (\pi, \{id_{X_i}\}_{(i,j) \in \bar{I} \times \bar{J}}) \\
 \pi^*(\{f_i\}_{i \in \bar{I}}) &= \{f_{(i,j)}\}_{(i,j) \in \bar{I} \times \bar{J}} \\
 \delta^*(\{X_{(i,j,z)}\}_{(i,j,z) \in (\bar{I} \times \bar{J}) \times \bar{J}}) &= \{X_{(i,j)}\}_{(i,j) \in \bar{I} \times \bar{J}} \\
 \bar{\delta}(\{X_{(i,j,z)}\}_{(i,j,z) \in (\bar{I} \times \bar{J}) \times \bar{J}}) &= (\delta, \{id_{X_{(i,j,z)}}\}_{(i,j) \in \bar{I} \times \bar{J}}) \\
 \delta^*(\{f_{(i,j,z)}\}_{(i,j,z) \in (\bar{I} \times \bar{J}) \times \bar{J}}) &= \{f_{(i,j)}\}_{(i,j) \in \bar{I} \times \bar{J}}
 \end{aligned}$$

codomain fibration

$$\begin{array}{ccc}
 \bullet \longrightarrow X & X \xrightarrow{f} Y & \pi^* \varphi \\
 \pi^*(\varphi) \downarrow \ulcorner & \downarrow \varphi & \downarrow \psi \\
 I \times J \xrightarrow{\pi} I & I & I \times J \rightarrow \bar{I}
 \end{array}
 \Rightarrow
 \begin{array}{ccc}
 \bullet & \xrightarrow{\quad} & X \\
 \downarrow \ulcorner & \swarrow \pi^*(f) & \downarrow f \\
 \bullet & \xrightarrow{\quad} & Y \\
 \downarrow \ulcorner & \swarrow \pi^*(\varphi) & \downarrow \psi \\
 \bullet & \xrightarrow{\quad} & I \times J \rightarrow \bar{I}
 \end{array}$$

$$\begin{array}{ccc}
 \bullet \longrightarrow X & X \xrightarrow{f} Y & \delta^*(\varphi) \\
 \delta^*(\varphi) \downarrow \ulcorner & \downarrow \varphi & \downarrow \psi \\
 I \times J \xrightarrow{\delta} (I \times J) \times J & (I \times J) \times J & (I \times J) \times J \\
 \delta^*(\varphi) \downarrow \ulcorner & \swarrow \delta^*(f) & \downarrow f \\
 \bullet & \xrightarrow{\quad} & Y \\
 \downarrow \ulcorner & \swarrow \delta^*(\varphi) & \downarrow \psi \\
 \bullet & \xrightarrow{\quad} & (I \times J) \times J
 \end{array}$$

subobject fibration と同様

# Exercise 1.4.1 続き

simple fibrationの場合

$$\pi^*(I, X) = (I \times J, X)$$

$$\bar{\pi}(I, X) = (\pi, \pi')$$

$$\pi^*(id_I, f) = (id_{I \times J}, f \circ (\pi \times id))$$

$$\pi^*(id_{(I, X)}) = \pi^*(id_I, \pi')$$

$$= (id_{I \times J}, \pi' \circ (\pi \times id))$$

$$= (id_{I \times J}, \pi') = id_{\pi^*(I, X)}$$

$$\pi^*(id_I, g \circ \langle \pi, f \rangle) = (id_{I \times J}, g \circ \langle \pi, f \rangle \circ (\pi \times id))$$

$$= (id_{I \times J}, g \circ \langle \pi \circ (\pi \times id), f \circ (\pi \times id) \rangle)$$

$$= (id_{I \times J}, g \circ (\pi \times id) \circ \langle \pi, f \circ (\pi \times id) \rangle)$$

$$= \pi^*(id_I, g) \circ \pi^*(id, f)$$

Exercise 1.4.2

Exercise 1.1.3 (ii) と同じ

# Exercise 1.4.3

$-^*$  と  $-'^*$  が共に cleavage だとする

$$\begin{array}{ccc}
 & & \bar{u}'(Y) \\
 & \searrow & \\
 u'^*(Y) & \xrightarrow{\alpha_Y} & u^*(Y) \xrightarrow{\bar{u}(f)} Y \\
 & & \bar{u}(f) \\
 I \equiv I & \longrightarrow & J
 \end{array}$$

terminal lifting  
 なのて:  $u^*(Y) \cong u'^*(Y)$

$$\begin{array}{ccc}
 u'^*(X) & \xrightarrow{\alpha_X} & u^*(X) \xrightarrow{\bar{u}(X)} X \\
 u^*(f) \downarrow & & u^*(f) \downarrow & & \downarrow f \\
 u'^*(Y) & \xrightarrow{\alpha_Y} & u^*(Y) \xrightarrow{\bar{u}(Y)} Y \\
 I \equiv I & \xrightarrow{u} & J
 \end{array}$$

$u^*(Y) \rightarrow Y$  が terminal lifting なのて  
 $f \circ \bar{u}(X) \circ \alpha_X$  が  $S$  の一意な射が存在

$u^*(f) \circ \alpha_X$  は

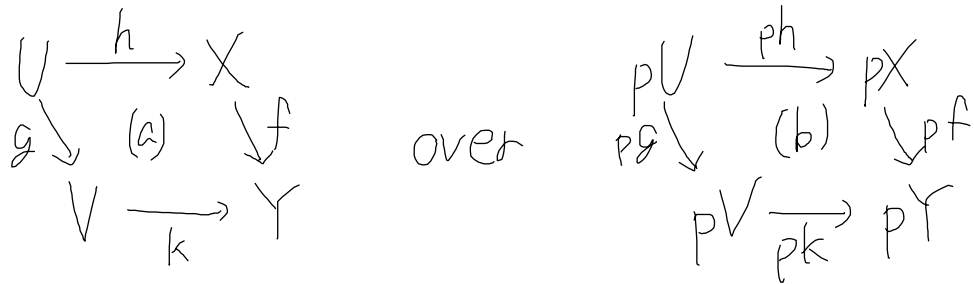
$$\begin{aligned}
 & \bar{u}(Y) \circ u^*(f) \circ \alpha_X \\
 &= f \circ \bar{u}(X) \circ \alpha_X \\
 & \text{より条件を満たす}
 \end{aligned}$$

ゆえに  $u^*(f) \circ \alpha_X = \alpha_Y \circ u'^*(f)$  ぞ  
 $\alpha$  は natural

$\alpha_Y \circ u'^*(f)$  は

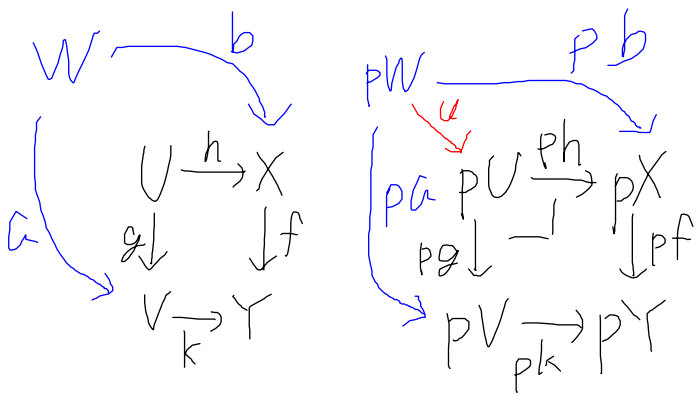
$$\begin{aligned}
 & \bar{u}(Y) \circ \alpha_Y \circ u'^*(f) \\
 &= \bar{u}'(Y) \circ u'^*(f) \\
 &= f \circ \bar{u}'(X) = f \circ \bar{u}(X) \circ \alpha_X \\
 & \text{より条件を満たす。}
 \end{aligned}$$

# Exercise 1.4.4 (if)

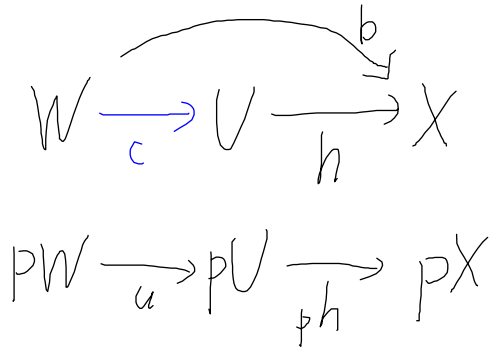


$h$  and  $k$  are cartesian.

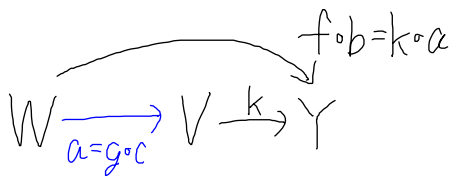
(a) is PB  $\Leftrightarrow$  (b) is PB



$h$  が "cartesian" なので



$k$  が "cartesian" なので



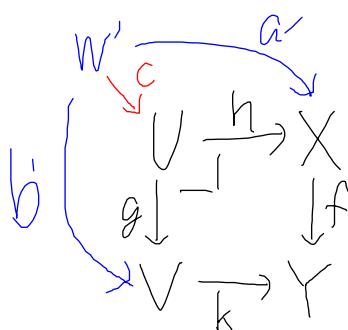
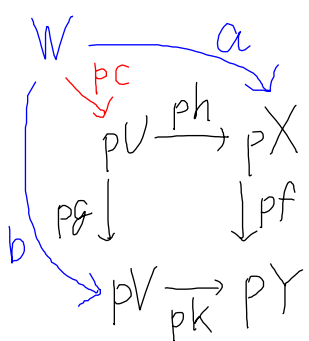
条件を満たす  $W \rightarrow U$  は (b) が "PB" なので above  $u$  での [図] から一意.

$$\begin{array}{ccc}
 pW & \xrightarrow{pa} & pV & \xrightarrow{pk} & pY \\
 & = pg \circ u & & & 
 \end{array}$$

# Exercise 1.4.4 (only if)



(a) is PB  $\Rightarrow$  (b) is PB



let  $W' \xrightarrow{a'} X$  and  $W' \xrightarrow{b'} V$  be cartesian lifting of  $a$  and  $b$  resp.

$$\begin{aligned}
 ph \circ pc &= p(h \circ c) = pa' = a \\
 pg \circ pc &= p(g \circ c) = pb' = b
 \end{aligned}$$

$W \xrightarrow{d} pU$  が条件を満たすと  
 $W' \xrightarrow{d'} U$  above  $d$  が上図を可換にするか  
 (a) は PB なのて  $d' = c$  て  $d = pc$   
 よて一意



Exercise 1.4.5 every poset fibration is split

$$\begin{array}{ccc}
 & u^*v^*X \rightarrow v^*X & \\
 \cong \downarrow & & \searrow \\
 \mathbb{E} & (v \circ u)^*X & \rightarrow X \\
 \downarrow P & & \\
 \mathbb{B} & & 
 \end{array}$$

$$I \xrightarrow{u} J \xrightarrow{v} K$$

$\mathbb{E}_I$  が poset なので  $u^*v^*X = (v \circ u)^*X$

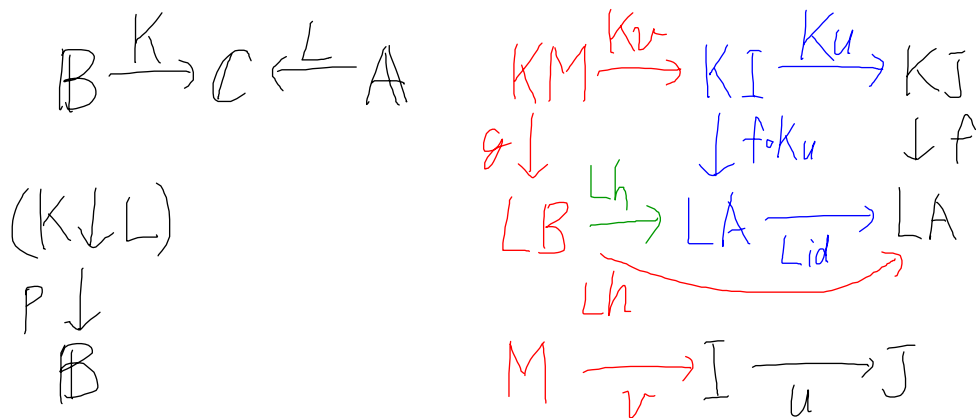
$$\begin{array}{ccc}
 u^*v^*X & \xlongequal{\quad} & (v \circ u)^*X \\
 u^*v^*f \downarrow & & \downarrow (v \circ u)^*f \\
 u^*v^*Y & \xlongequal{\quad} & (v \circ u)^*Y
 \end{array}$$

$u^*v^*(-) \simeq (v \circ u)^*$  は自然同型

より、 $u^*v^*f = (v \circ u)^*f$

identity に関しても同様

# Exercise 1.4.6



$$\begin{aligned}
 u^*(J, A, f) &= (I, A, f \circ Ku) \\
 \bar{u}(J, A, f) &= (u, id)
 \end{aligned}$$

$$id^*(J, A, f) = (J, A, f \circ Kid) = (J, A, f)$$

$$\begin{aligned}
 v^*u^*(J, A, f) &= v^*(I, A, f \circ Ku) \\
 &= (M, A, f \circ Ku \circ Kv) \\
 &= (M, A, f \circ K(u \circ v)) \\
 &= (u \circ v)^*(J, A, f)
 \end{aligned}$$

$\mathcal{L} \circ \tau$  split fibration

# Exercise 1.4.7

Fam( $\mathbb{C}$ )

↓

Sets

$$\begin{array}{ccc}
 \{X_{u(j)}\}_{j \in J} & \xrightarrow{\{id_{X_{u(i)}}\}_{i \in I}} & \{X_i\}_{i \in I} \\
 \{f_{u(j)}\}_{j \in J} \downarrow & & \downarrow \{f_i\}_{i \in I} \\
 \{Y_{u(j)}\}_{j \in J} & \xrightarrow{\{id_{Y_{u(i)}}\}_{i \in I}} & \{Y_i\}_{i \in I}
 \end{array}$$

$$J \xrightarrow{u} I$$

$$\bar{\Psi} : \text{Sets}^{\text{op}} \rightarrow \text{Cats}$$

$$\bar{\Psi}(I) = \mathbb{C}^I$$

$$\begin{aligned}
 \bar{\Psi}(u) : \mathbb{C}^I &\rightarrow \mathbb{C}^J \text{ for } J \xrightarrow{u} I \\
 &\text{in Sets} \\
 \bar{\Psi}(u)(\{X_i\}_{i \in I}) &= \{X_{u(j)}\}_{j \in J} \\
 \bar{\Psi}(u)(\{f_i\}_{i \in I}) &= \{f_{u(j)}\}_{j \in J}
 \end{aligned}$$

# Exercise 1.4.8

$$\begin{array}{ccc}
 (B \downarrow p) = E \times_B B^{\rightarrow} & \longrightarrow & B^{\rightarrow} \\
 p^*(\text{cod}) \downarrow & & \downarrow \text{cod} \\
 E & \xrightarrow{p} & B
 \end{array}$$

$$\begin{aligned}
 (E \times_B B^{\rightarrow}) & ((X, u: I \rightarrow pX), \\
 & \cong \{ (f: X \rightarrow Y, w: (Y, v: J \rightarrow pY)) \\
 & \quad | v \circ w = p f \circ u \}
 \end{aligned}$$

in Cats

$$\begin{aligned}
 F: E^{\rightarrow} & \rightarrow E \times_B B^{\rightarrow} \\
 (f: X' \rightarrow X) & \mapsto (X, pf)
 \end{aligned}$$

$$\begin{array}{ccc}
 & & \text{y of} \\
 & \searrow & \downarrow \\
 X' & \xrightarrow{h} & u^*(Y) \xrightarrow{\bar{u}(Y)} Y
 \end{array}$$

$p$  is cloven fibration  $\text{よ} \text{う} \text{に} \text{し} \text{て}$

$$G: E \times_B B^{\rightarrow} \rightarrow E^{\rightarrow}$$

$$G(X, u: I \rightarrow pX) = \bar{u}(X) \text{ とお} \text{く}$$

$$pX' \xrightarrow{v} I \xrightarrow{u} pY$$

$$\begin{aligned}
 (E \times_B B^{\rightarrow}) (Ff, (Y, u: I \rightarrow pY)) & \cong \{ (g: X \rightarrow Y, v: pX' \rightarrow I) \mid u \circ v = p g \circ p f \} \\
 & \cong \{ (g: X \rightarrow Y, h: X' \rightarrow u^*(Y)) \mid \bar{u}(Y) \circ h = g \circ f \} \\
 & \cong E^{\rightarrow}(f, \bar{u}(Y)) = E^{\rightarrow}(f, G(Y, u))
 \end{aligned}$$

$$FG(X, u) = F \bar{u}(X) = (X, p \bar{u}(X)) = (X, u)$$

$FG \Rightarrow Id$  は counit である事は次ページで確認する。

$G: E \times_B B^{\rightarrow} \rightarrow E^{\rightarrow}$  を  $F$  の right-inverse right-adjoint とする

$$F_x: E/X \rightarrow B/pX$$

$$F_x = p$$

$$G_x: B/pX \rightarrow E/X$$

$$G_x = G(X, -)$$

とおく

$$\begin{aligned}
 FG = Id \text{ であること} & \text{ cod}(G(X, u)) = X \\
 & \text{かつ } p(G(X, u)) = u
 \end{aligned}$$

$\bar{u}(X) : u^*(X) \rightarrow X$  を  $G_X(u)$  で定義

$$Y \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} u^*(X) \rightarrow X$$

$$K \xrightarrow{v} J \xrightarrow{u} I$$

$$\begin{aligned} & \mathbb{B}/pX(pf, u) \\ &= \mathbb{B}/pX(F_X f, u) \\ &\cong \mathbb{E}/X(f, G_X(u)) \\ &= \mathbb{E}/X(f, \bar{u}(X)) \end{aligned}$$

$$\begin{array}{ccc} K \xrightarrow{v} J & & Y \xrightarrow{g} u^*(X) \\ pf \searrow & \Leftrightarrow & f \searrow \\ & & X \\ & & \swarrow \bar{u}(X) \\ & & I \end{array}$$

in  $\mathbb{B}/pX$   in  $\mathbb{E}/X$

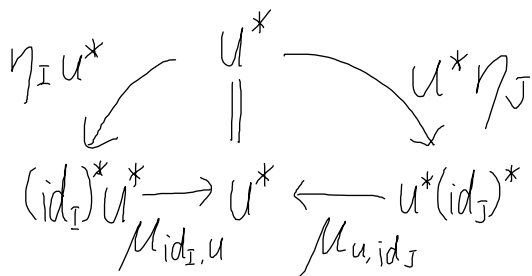
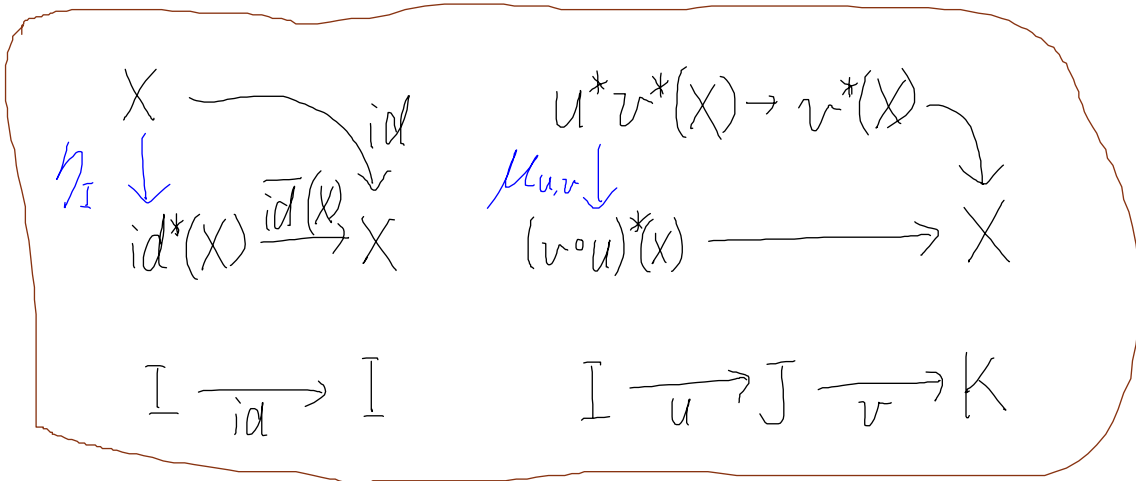
$$v = \varepsilon_f \circ F_X(g) = pg \text{ なのて } g \text{ は above } v$$

$\therefore G$  は cleavage

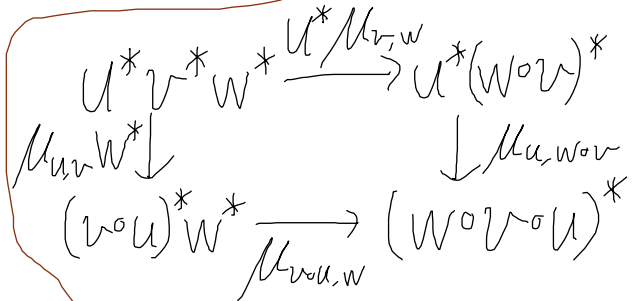
$FG \Rightarrow Id$  が counit であることの確認

$$\begin{array}{ccc} u^*(Y) \xrightarrow{id} u^*(Y) & & pu^*(Y) \xrightarrow{id} I \\ \bar{u}(Y) \downarrow & \Leftrightarrow & p\bar{u}(Y) \downarrow \\ Y \xrightarrow{id} Y & \text{in } \mathbb{E} & pY \xrightarrow{pid} pY \\ & & \downarrow u \\ & & Y \xrightarrow{id} Y \\ id_{G(Y,u)} \in E^{\rightarrow}(G(Y,u), G(Y,u)) & & \varepsilon_{(Y,u)} \in (E \times_B B^{\rightarrow})(FG(Y,u), (Y,u)) \end{array}$$

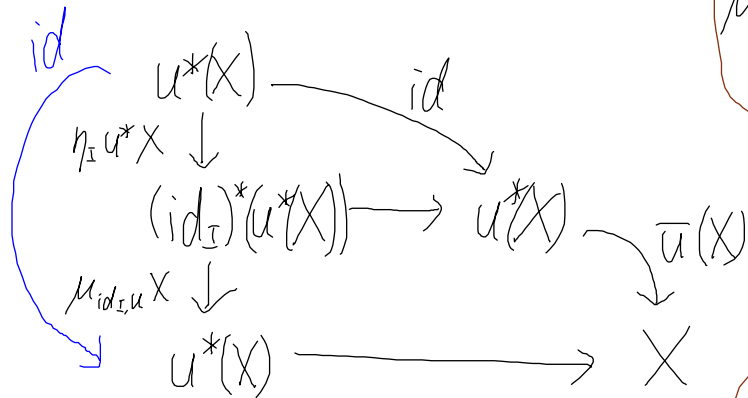
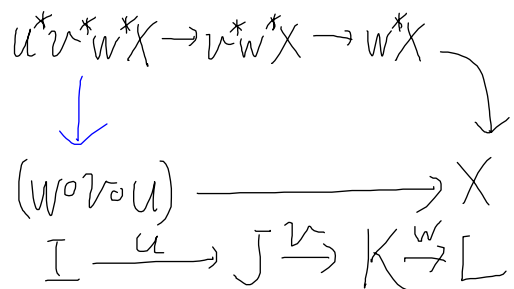
# Exercise 1.4.9



for  $u: I \rightarrow J$



for  $I \xrightarrow{u} J \xrightarrow{v} K \xrightarrow{w} L$



よって  $\mu_{id_I, u} \circ \eta_I u^* = id$   
 同様にして  $\mu_{u, id_I} \circ u^* \eta_J = id$

$\mu_{v∘u, w} \circ \mu_{u, w}$  と  $\mu_{u, w∘v} \circ \mu_{v, w}$  は共に上図を可換にするので等しい。

# Exercise 1.4.10

Indexed category  $\Psi: \mathcal{B}^{op} \rightarrow \text{Cat}$

Category  $\int \Psi$

object  $(I, X)$  where  $I \in |\mathcal{B}|, X \in |\Psi(I)|$

morphism  $(u, f): (I, X) \rightarrow (J, Y)$

where  $u \in \mathcal{B}(I, J)$

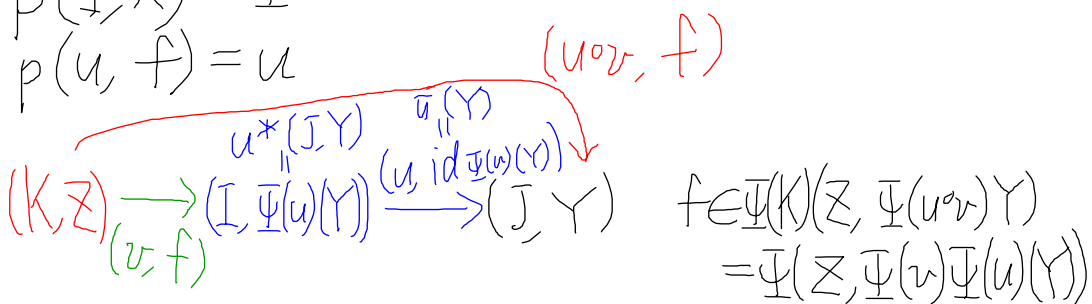
$f \in \Psi(I)(X, \Psi(u)(Y))$

composition  $(I, X) \xrightarrow{(u, f)} (J, Y) \xrightarrow{(v, g)} (K, Z)$   
 $= (v \circ u, \Psi(u)(g) \circ f)$

$p: \int \Psi \rightarrow \mathcal{B}$

$p(I, X) = I$

$p(u, f) = u$



$$K \xrightarrow{v} I \xrightarrow{u} J$$

$\Leftarrow \int p$  is cloven fibration

$$\text{id}^*(J, Y) = (J, \Psi(\text{id})(Y)) = (J, Y)$$

$$(u \circ v)^*(J, Y) = (J, \Psi(u \circ v)(Y)) = (J, \Psi(v)\Psi(u)(Y)) = v^*u^*(J, Y)$$

$\Leftarrow \int p$  is split