A Note on Freyd Dialgebra

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1 Preliminaly

Definitions are omitted in the meantime. Please consult Freyd[1].

2 Main result

Theorem 1. *Suppose T is a bifunctor on CPO-category such that map $Tf'f : TAA \to TBB$ is strict for arbitrary $A, B, f : A \to B, f' : B \to A$. And suppose (F, ϕ) is a special invariant object. Then (F, ϕ, ϕ^{-1}) is a free T-dialgebra.*

Proof. Let (A, ψ, ψ') be a T-dialgebra.

For the existence of dialgebra-map from (F, ϕ, ϕ^{-1}) to (A, ψ, ψ'), let $b_n : F \to A$ and $b'_n : A \to F$ be

$$\begin{aligned} (b_0, b'_0) &= (\psi \circ \bot, \bot) \\ (b_{n+1}, b'_{n+1}) &= (\psi \circ Tb'_n b_n \circ \phi^{-1} \\ &\quad , \phi \circ Tb_n b'_n \circ \psi') \end{aligned}$$

. Then $\{(b_n, b'_n)\}_{n<\omega}$ forms acsending chain and has the least upper bound $(b, b') = \bigsqcup_{n<\omega}(b_n, b'_n)$ which is a required dialgebras-map.

For the uniquness. Suppose $(x, x') : (F, \phi, \phi^{-1}) \to (X, \psi, \psi')$ be another dialgebra-map. Let $e_n : F \to F$ be acsending chain of idempotents defined by $e_0 = \bot$, $e_{n+1} = \phi \circ Te_n e_n \circ \phi^{-1}$.

We show that $x \circ e_n = b_n$ and $e_n \circ x' = b'_n$ by induction.

For the base case

$$\begin{aligned} x \circ e_0 &= (\psi \circ Tx'x \circ \phi^{-1}) \circ \bot \\ &= \{\text{isomorphism } \phi^{-1} \text{ is strict}\} \\ &\quad \psi \circ Tx'x \circ \bot \\ &= \{Tx'x \text{ is strict by assumption}\} \\ &\quad \psi \circ \bot \\ &= b_0 \end{aligned}$$

$$\begin{aligned} e_0 \circ x' &= \bot \circ x' \\ &= \bot \\ &= b'_0 \end{aligned}$$

For the induction step, assuming $x \circ e_n = b_n$ and $e_n \circ x' = b'_n$,

$$\begin{aligned} x \circ e_{n+1} &= (\psi \circ Tx'x \circ \phi^{-1}) \circ (\phi \circ Te_n e_n \circ \phi^{-1}) \\ &= \psi \circ T(e_n \circ x')(x \circ e_n) \circ \phi^{-1} \\ &= \psi \circ Tb'_n b_n \circ \phi^{-1} \\ &= b_{n+1} \end{aligned}$$

$$\begin{aligned} e_{n+1} \circ x' &= (\phi \circ Te_n e_n \phi^{-1}) \circ (\phi \circ Txx' \circ \psi') \\ &= \phi \circ T(x \circ e_n)(e_n \circ x') \circ \psi' \\ &= \phi \circ Tb_n b'_n \circ \psi' \\ &= b'_{n+1} \end{aligned}$$

By induction, $x \circ e_n = b_n$ and $e_n \circ x' = b'_n$ hold for all $n < \omega$.

Finally

$$\begin{aligned}
(x, x') &= (x \circ 1_F, 1_F \circ x') \\
&= \{\ (F, \phi) \text{ is a special invariant object }\} \\
&\quad (x \circ (\bigsqcup_{n<\omega} e_n), (\bigsqcup_{n<\omega} e_n) \circ x') \\
&= \{\text{composition is continuous}\} \\
&\quad (\bigsqcup_{n<\omega} (x \circ e_n), \bigsqcup_{n<\omega} (e_n \circ x')) \\
&= \{\text{pairing is continuous}\} \\
&\quad \bigsqcup_{n<\omega} (x \circ e_n, e_n \circ x') \\
&= \bigsqcup_{n<\omega} (b_n, b'_n) \\
&= (b, b')
\end{aligned}$$

□

Corollary 1. *If T is a covariant functor on CPO-category and Tf is always strict, then (F, ϕ) is an initial algebra iff (F, ϕ^{-1}) is a final coalgebra.*

References

[1] FREYD, P. J. 1990. Recursive Types Reduced to Inductive Types. In Proceedings 5th IEEE Annual Symp. on Logic in Computer Science, LICS'90 (Philadelphia, PA, USA, 4-7 June 1990). IEEE Computer Society Press, Los Alamitos, CA, 498-507. `http://ieeexplore.ieee.org/xpl/freeabs_all.jsp?arnumber=113772`