p.41

$s_{\mathbb{B}} : s(\mathbb{B}) \to \mathbb{B}$ がfibrationであることの確認

$$
\begin{tikzcd}
(K, Z) \arrow[r, "{(v, g)}"'] \arrow[rr, bend left, "{(u \circ v, f)}"] & (I, Y) \arrow[r, "{(u, \pi')}"] & (J, Y)
\end{tikzcd}
$$

$$K \xrightarrow[v]{} I \xrightarrow[u]{} J$$

$$(u, \pi') \circ (v, g) = (u \circ v, f)$$

$$\Leftrightarrow \quad g = f$$

$\therefore$ (v, f)は条件を満たす一意な射

$$\mathbb{B} \xrightarrow[\cong]{1^*} \mathbb{B}/\!/1$$

$$\mathbb{B}(X, Y) \cong \mathbb{B}(1 \times X, Y) = \mathbb{B}/\!/1(X, Y)$$

$$\mathrm{Ob}(\mathbb{B}) = \mathrm{Ob}(\mathbb{B}/\!/1)$$

$$\mathbb{B} \xrightarrow[\cong]{1^*} \mathbb{B}/1$$

$$\mathbb{B}(X, Y) \cong \mathbb{B}/1(X, Y)$$

$$\mathrm{Ob}(\mathbb{B}) \cong \mathrm{Ob}(\mathbb{B}/1)$$

P.43

A pullback of a mono along an arbitrary map is mono again.

$$
\begin{tikzcd}
Z \arrow[dr, dashed, "h=h'"] \arrow[drr, bend left, "p\circ h=p\circ h'"] \arrow[ddr, bend right, "g\circ h=g\circ h'"'] & & \\
& Y \arrow[r, "p"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, hook, "f"] \\
& J \arrow[r, "u"'] & I
\end{tikzcd}
$$

$h, h' : Z \to Y$

$g \circ h = g \circ h'$ とする

$$
\begin{aligned}
f \circ p \circ h &= u \circ g \circ h \\
&= u \circ g \circ h' \\
&= f \circ p \circ h'
\end{aligned}
$$

fがmonoなので

$p \circ h = p \circ h'$

よってh, h'は左上の図式を可換にする一意な射

$\therefore\ h = h'$

$\therefore\ g$はmono

Exercise 1.3.1 (i)

$$
\begin{tikzcd}
s(\mathbb{B}) \arrow[d, "s_{\mathbb{B}}"'] & (K,z) \arrow[r, "{(v,i)}"] \arrow[rr, bend left, "{(u\circ v,\ g)}"] & (I,X) \arrow[r, "{(u,f)}"] & (J,Y) \\
\mathbb{B} & K \arrow[r, "v"'] & I \arrow[r, "u"'] & J
\end{tikzcd}
$$

$$
\begin{aligned}
&(u,f)\circ(v,i) \\
&= (u\circ v,\ f\circ\langle v\circ\pi, i\rangle)
\end{aligned}
$$

$$
\begin{tikzcd}
 & I & \\
I\times X \arrow[r, "h"', "\cong"] \arrow[ur, bend left, "\pi"] \arrow[dr, bend right, "f"'] & I\times Y \arrow[u, "\pi"'] \arrow[d, "\pi'"] & \\
 & Y &
\end{tikzcd}
\Rightarrow
\begin{aligned}
&f\circ\langle v\circ\pi, i\rangle = g \\
&\langle v\circ\pi, f\circ\langle v\circ\pi, i\rangle\rangle = \langle v\circ\pi, g\rangle \\
&\langle \pi\circ\langle v\circ\pi, i\rangle, f\circ\langle v\circ\pi, i\rangle\rangle = \langle v\circ\pi, g\rangle \\
&\langle \pi, f\rangle\circ\langle v\circ\pi, i\rangle = \langle v\circ\pi, g\rangle \\
&\langle v\circ\pi, i\rangle = h^{-1}\circ\langle v\circ\pi, g\rangle \\
&i = \pi'\circ h^{-1}\circ\langle v\circ\pi, g\rangle
\end{aligned}
$$

よって(u,f)はcartesian

assume (u,f) is cartesian

$$
\begin{tikzcd}
(I,Y) \arrow[r, "{(id,i)}"] \arrow[rr, bend left, "{(u,\pi')}"] & (I,X) \arrow[r, "{(u,f)}"] & (J,Y) \\
I \arrow[r, "id"'] & I \arrow[r, "u"'] & J
\end{tikzcd}
$$

$i : I\times Y \to X$ such that $f\circ\langle \pi, i\rangle = \pi'$

$$
\begin{aligned}
&\langle \pi, f\rangle\circ\langle \pi, i\rangle \\
&= \langle \pi\circ\langle \pi, i\rangle, f\circ\langle \pi, i\rangle\rangle \\
&= \langle \pi, \pi'\rangle = id_{I\times Y}
\end{aligned}
\qquad
\begin{aligned}
&\langle \pi, i\rangle\circ\langle \pi, f\rangle \\
&= \langle \pi\circ\langle \pi, f\rangle, i\circ\langle \pi, f\rangle\rangle \\
&= \langle \pi, \pi'\rangle = id_{I\times X}
\end{aligned}
$$

省略

Exercise 1.3.1 (ii)

$s(\mathbb{B}) \qquad \mathbb{B}^{\to}$

$$\begin{tikzcd} (I,X) \arrow[d, "{(u,f)}"'] & \mapsto & I^*(X) = \left(\begin{smallmatrix} I\times X \\ \downarrow \pi \\ I \end{smallmatrix}\right) \arrow[d, bend left, "u^* f"] \\ (J,Y) & \mapsto & J^*(Y) = \left(\begin{smallmatrix} J\times Y \\ \downarrow \pi \\ J \end{smallmatrix}\right) \end{tikzcd}$$

$$\begin{tikzcd} I\times X \arrow[r, "{\langle u\circ\pi, f\rangle}"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft \quad u^*(f)"] & J\times Y \arrow[d, "\pi"] \\ I \arrow[r, "u"'] & J \end{tikzcd}$$

$u: I \to J$
$f: I\times X \to Y$

id の保存

$$(I,X) \xrightarrow[\substack{id_{(I,X)} \\ =(id_I, \pi')}]{} (I,X) \quad \longmapsto \quad \begin{tikzcd} I\times X \arrow[r, "{\langle \pi, \pi' \rangle = id}"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft \quad id_I^* \pi'"] & I\times X \arrow[d, "\pi"] \\ I \arrow[r, "id"'] & I \end{tikzcd}$$

composition の保存

$$\begin{tikzcd} (I,X) \arrow[r, "{(u,f)}"] \arrow[rr, bend right, "{(v\circ u,\, g\circ\langle u\circ\pi, f\rangle)}"'] & (J,Y) \arrow[r, "{(v,g)}"] & (K,Z) \end{tikzcd}$$

$$\longmapsto \begin{tikzcd} I\times X \arrow[r, "{\langle u\circ\pi, f\rangle}"] \arrow[d, "\pi"'] & J\times Y \arrow[r, "{\langle v\circ\pi, g\rangle}"] \arrow[d, "\pi"] & K\times Z \arrow[d, "\pi"] \\ I \arrow[r, "u"'] & J \arrow[r, "v"'] & K \end{tikzcd}$$

$$\longmapsto \begin{tikzcd} I\times X \arrow[r, "{\langle v\circ u\circ\pi,\, g\circ\langle u\circ\pi, f\rangle\rangle}"] \arrow[d, "\pi"'] & K\times Z \arrow[d, "\pi"] \\ I \arrow[r, "v\circ u"'] & K \end{tikzcd}$$

$$\langle v\circ\pi, g\rangle \circ \langle u\circ\pi, f\rangle$$
$$= \langle v\circ u\circ\pi,\, g\circ\langle u\circ\pi, f\rangle\rangle$$

続く

Exercise 1.3.1 (ii) 続き

$(u,f):(I,X)\longrightarrow(J,Y)$ が cartesian とする

$$
\begin{tikzcd}
Z \arrow[dr, "x"] \arrow[drr, bend left, "b"] \arrow[ddr, bend right, "a"'] & & \\
& I\times X \arrow[r, "\langle u\circ\pi, f\rangle"] \arrow[d, "\pi"'] & J\times Y \arrow[d, "\pi"] \\
& I \arrow[r, "u"'] & J
\end{tikzcd}
\qquad
\begin{tikzcd}
& I & \\
I\times X \arrow[ur, "\pi"] \arrow[r, "h"] \arrow[dr, "f"'] & I\times Y \arrow[u, "\pi"'] \arrow[d, "\pi'"] \\
& Y
\end{tikzcd}
$$

$u\circ a=\pi\circ b$ を満たす a,b が与えられたとする

$$
\boxed{
\begin{aligned}
&\pi\circ x=a \wedge f\circ x=\pi'\circ b\\
&\Rightarrow \langle u\circ\pi, f\rangle\circ x\\
&\quad=\langle u\circ\pi\circ x, f\circ x\rangle\\
&\quad=\langle u\circ a, \pi'\circ b\rangle\\
&\quad=\langle \pi\circ b, \pi'\circ b\rangle\\
&\quad=b
\end{aligned}
}
$$

$$
\begin{pmatrix}
\pi\circ x=a\\
\langle u\circ\pi, f\rangle\circ x=b
\end{pmatrix}
$$

$$
\Leftrightarrow
\left(
\begin{aligned}
&\pi\circ x=a\\
&f\circ x=\pi'\circ b
\end{aligned}
\right.
$$

$$
\Leftrightarrow h\circ x=\langle\pi, f\rangle\circ x=\langle a, \pi'\circ b\rangle
$$

$$
\Leftrightarrow x=h^{-1}\circ\langle a, \pi'\circ b\rangle
$$

よって条件を満たす x が一意に存在

ゆえに u^*f は pullback square

Exercise 1.3.2 (i) $\mathbb{B}$がfinite productを持つとする

$\mathbb{B}/\!/I$がfinite productを持つことを示す

$\mathbb{B}(I\times X, 1)$
$\cong \mathbb{B}/I(X, 1)$

$\mathbb{B}/\!/I(Z, X)\times\mathbb{B}/\!/I(Z, Y)$
$\cong \mathbb{B}(I\times Z, X)\times\mathbb{B}(I\times Z, Y)$
$\cong \mathbb{B}(I\times Z, X\times Y)$
$\cong \mathbb{B}/\!/I(Z, X\times Y)$

$I^*: \mathbb{B}\to\mathbb{B}/I$ がfinite productを保存することは明らか

Exercise 1.3.2 (ii) $\mathbb{B}$ が finite product を持つとする

(a) $\mathbb{B}$ is cartesian-closed
(b) each fibre $\mathbb{B}//I$ is cartesian closed
(c) each functor $I^*: \mathbb{B} \to \mathbb{B}//I$ has a right adjoint $I \Rightarrow (-)$

(a) ⇒ (b)

terminal object (i)で示した
product (i)で示した
exponential $\mathbb{B}//I(X \times Y, Z)$
$= \mathbb{B}(I \times X \times Y, Z)$
$= \mathbb{B}(I \times X, Y \Rightarrow Z)$
$= \mathbb{B}//I(X, Y \Rightarrow Z)$

(b) ⇒ (a)

仮定より $\mathbb{B}$ に terminal object と product はある
exponential は $\mathbb{B}(X \times Y, Z) \cong \mathbb{B}//X(Y, Z) \cong \mathbb{B}//X(1, Y \Rightarrow Z)$
$\cong \mathbb{B}(X \times 1, Y \Rightarrow Z) \cong \mathbb{B}(X, Y \Rightarrow Z)$

(a) ⇒ (c)

$\mathbb{B}//I(I^*(X), Y) \cong \mathbb{B}//I(X, Y) \cong \mathbb{B}(I \times X, Y) \cong \mathbb{B}(X, I \Rightarrow Y)$

(c) ⇒ (a)

仮定より $\mathbb{B}$ に terminal object と product はある
exponential $\mathbb{B}(X \times Y, Z) \cong \mathbb{B}//Y(Y^*(X), Z) \cong \mathbb{B}(X, Y \Rightarrow Z)$

Exercise 1.3.3　$\mathbb{B}$ has finite limit

各 $I^*: \mathbb{B} \to \mathbb{B}/I$ が right adjoint を持つ $\Leftarrow$ $\mathbb{B}$ は CCC

$\varphi: X \to I$ に対して equalizer

$$\begin{tikzcd} \Pi_I(\varphi) \arrow[r, tail, "e"] & (I \Rightarrow X) \arrow[r, shift left, bend left, "I \Rightarrow \varphi"] \arrow[r, shift right, bend right, "\Lambda(\pi')"'] & (I \Rightarrow I) \end{tikzcd}$$

$$\begin{tikzcd} I \times Y \arrow[r, "f"] \arrow[dr, "\pi"'] & X \arrow[d, "\varphi"] \\ & I \end{tikzcd}$$

の $\Pi_I(\varphi)$ が条件を満たすことを示す

$\mathbb{B}/I(I^*Y, \varphi)$

$\cong \{ f: I \times Y \to X \mid \pi = \varphi f \}$

$\cong \{ g: Y \times I \to X \mid \pi' = \varphi g \}$

$\cong \{ h: Y \to [I \Rightarrow X] \mid \Lambda(\pi') = (I \Rightarrow \varphi) \circ h \}$

$\cong \{ h: Y \to [I \Rightarrow X] \mid \Lambda(\pi') \circ h = (I \Rightarrow \varphi) \circ h \}$

$\cong \mathbb{B}(Y, \Pi_I \varphi)$

各 $I^*: \mathbb{B} \to \mathbb{B}/I$ が right adjoint を持つ

$\Rightarrow$ $\mathbb{B}$ は CCC

$X \Rightarrow Y \doteqdot \Pi_X X^* Y$ が条件を満たす

$\mathbb{B}(Z, \Pi_X X^* Y)$

$= \mathbb{B}/X(X^* Z, X^* Y)$

$= \{ f: X \times Z \to X \times Y \mid \pi \circ f = \pi \}$

$= \{ g: X \times Z \to Y \}$

$= \mathbb{B}(X \times Z, Y)$

$= \mathbb{B}(Z \times X, Y)$

Exercise 1.3.4 (i)

comonad $(I\times(-), \varepsilon, \delta)$

with $\varepsilon_X \triangleq \pi' : I\times X \to X$

$\delta_X \triangleq \langle \pi, id_{I\times X}\rangle : I\times X \to I\times I\times X$

$$
\begin{tikzcd}
I\times I\times I\times X & I\times I\times X \arrow[l, "I\times\delta_X"'] \\
I\times I\times X \arrow[u, "\delta_{I\times X}"] & I\times X \arrow[l, "\delta_X"] \arrow[u, "\delta_X"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
I\times X & I\times I\times X \arrow[l, "I\times\varepsilon_X"'] \arrow[r, "\varepsilon_{I\times X}"] & I\times X \\
& I\times X \arrow[u, "\delta_X"'] \arrow[ul, bend left] \arrow[ur, bend right] &
\end{tikzcd}
$$

2つの図式は明らかに可換

Exercis 1.3.4 (ii)

comonad $(I\times(-):\mathbb{B}\to\mathbb{B}, \varepsilon, \delta)$ の coKleisli category

Objects $X, Y \in Ob(\mathbb{B})$

Morphisms $hom(X,Y) \to \mathbb{B}(I\times X \to Y)$

$id_X \in hom(X,X)$

$= \varepsilon_X \in \mathbb{B}(I\times X, X)$

$= \pi'$

$f\circ g = f\circ_{\mathbb{B}}(I\times g)\circ\delta$

$= f\circ_{\mathbb{B}}(I\times g)\circ\langle\pi, id\rangle$

$= f\circ_{\mathbb{B}}\langle\pi, g\rangle$

よって $\mathbb{B}//I$ と等しい。

comonad $(I\times(-), \varepsilon, \delta)$ の Eilenberg-Moore category

Objects $(X, \varphi: X\to I\times X)$

such that $\varepsilon\circ\varphi = id_X$

$\Leftrightarrow$ $\pi\circ\varphi: X\to I$ に対応 in $\mathbb{B}/I$

Morphisms $(X,\varphi)\xrightarrow{h}(Y,\psi)$ $\psi\circ h = (I\times h)\circ\varphi$

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "\varphi"'] & Y \arrow[d, "\psi"] \\ I\times X \arrow[r, "I\times h"'] & I\times Y \end{tikzcd}$$

$\Leftrightarrow \pi\psi h = \pi\varphi$

よって $\mathbb{B}/I$ と等しい。

Exercise 1.3.5

$$\begin{tikzcd} \begin{pmatrix} Z \\ \downarrow \\ K \end{pmatrix} \arrow[r, shift left, "{(g,v)}"] \arrow[r, shift right, "{(h,w)}"'] & \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix} \arrow[r, "{(f,u)}"] & \begin{pmatrix} Y \\ \downarrow \\ J \end{pmatrix} \end{tikzcd}$$

f, u が mono とすると

$$(f,u)(g,v) = (f,u)(h,w)$$
$$\Leftrightarrow (fg, uv) = (fh, uw)$$
$$\Leftrightarrow fg = fh \land uv = uw$$
$$\Leftrightarrow g = h \land v = w$$
$$\Leftrightarrow (g,v) = (h,w)$$

よって (f,u) は mono

(f,u) が mono とすると

$$\begin{tikzcd} Z \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] \arrow[d] & X \arrow[r, "f"] \arrow[d, "\varphi"'] & Y \arrow[d] \\ Z \arrow[r, shift left, "\varphi\circ g"] \arrow[r, shift right, "\varphi\circ h"'] & I \arrow[r, "u"'] & J \end{tikzcd}$$

$$fg = fh$$
$$\Leftrightarrow (f,u)\circ(g, \varphi\circ g) = (f,u)\circ(h, \varphi\circ h)$$
$$\Leftrightarrow (g, \varphi\circ g) = (h, \varphi\circ h)$$
$$\Leftrightarrow g = h$$

よって f は mono

Z は limit として定義

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d] & X \arrow[r, "f"] \arrow[d, "\varphi"] & Y \arrow[d] \\ K \arrow[r, shift left, "v"] \arrow[r, shift right, "w"'] & I \arrow[r, "u"'] & J \end{tikzcd}$$

$$u\circ v = u\circ w$$
$$\Leftrightarrow (f,u)\circ(g,v) = (f,u)\circ(g,w)$$
$$\Leftrightarrow (g,v) = (g,w) \Leftrightarrow v = w$$

よって u は mono

Exercise 1.3.6

$$
\begin{tikzcd}
Z \arrow[dr, "i"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "g"'] & & \\
& Y \arrow[r, "f"] \arrow[d, "\psi"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\varphi"] \\
& J \arrow[r, "u"'] & I \arrow[d, shift right, "a"'] \arrow[d, shift left, "b"] \\
& & A
\end{tikzcd}
$$

φ が regular mono で、
a, b の equalizer だとする

$a \circ u \circ g = b \circ u \circ g$

$\Rightarrow$ ─ φ が equalizer なので ─

$\exists ! h : Z \to X,\ \varphi \circ h = ug$

$\Rightarrow$ ─ PB ─

$\exists ! i : Z \to Y,\ \psi \circ i = g \wedge f \circ i = h$

$\therefore$ ψ は $a \circ u, b \circ u$ の equalizer であり regular mono

問題文には書いていないけれど、$\mathcal{B}$ が PB を持つことは仮定して良いんだよね？

Exercise 1.3.7 (i)

$\mathbb{B}$ が finite limit を持つとする

$\mathbb{B}$ が CCC $\Rightarrow$ $\mathbb{B}^{\to}$ は CCC

terminal object
$$\begin{tikzcd} X \arrow[r, "!"] \arrow[d, "\varphi"'] & 1 \arrow[d, "!"] \\ I \arrow[r, "!"'] & 1 \end{tikzcd}$$

product
$$\begin{pmatrix} X \\ \downarrow \varphi \\ I \end{pmatrix} \times \begin{pmatrix} Y \\ \downarrow \psi \\ J \end{pmatrix} = \begin{pmatrix} X \times Y \\ \downarrow \varphi \times \psi \\ I \times J \end{pmatrix}$$

$$\begin{aligned} &\mathbb{B}^{\to}(r, \varphi \times \psi) \\ &= \{ (f : Z \to X \times Y, u : K \to I \times J) \mid (\varphi \times \psi) \circ f = u \circ r \} \\ &= \{ (f_1 : Z \to X, f_2 : Z \to Y, u_1 : K \to I, u_2 : K \to J) \mid \varphi \circ f_1 = u_1 \circ r, \psi \circ f_2 = u_2 \circ r \} \\ &= \mathbb{B}^{\to}(r, \varphi) \times \mathbb{B}^{\to}(r, \psi) \end{aligned}$$

exponent
$$\begin{pmatrix} X \\ \downarrow \varphi \\ I \end{pmatrix} \Rightarrow \begin{pmatrix} Y \\ \downarrow \psi \\ J \end{pmatrix}$$
$$\begin{tikzcd} U \arrow[r, "p"] \arrow[d, "\varphi \Rightarrow \psi"'] \arrow[dr, phantom, "\lrcorner", very near start] & (X \Rightarrow Y) \arrow[d, "X \Rightarrow \psi"] \\ (I \Rightarrow J) \arrow[r, "\varphi \Rightarrow J"'] & (X \Rightarrow J) \end{tikzcd}$$

$$\begin{aligned} &\mathbb{B}^{\to}(r \times \varphi, \psi) \\ &= \{ (f : Z \times X \to Y, u : K \times I \to J) \mid \psi \circ f = u \circ (r \times \varphi) \} \\ &\cong \{ (f' : Z \to (X \Rightarrow Y), u' : K \to (I \Rightarrow J)) \mid (X \Rightarrow \psi) \circ f' = (\varphi \Rightarrow J) \circ u' \circ r \} \\ &\cong \{ (g : Z \to U, u' : K \to (I \Rightarrow J)) \mid (\varphi \Rightarrow \psi) \circ g = u' \circ r \} \\ &\cong \mathbb{B}^{\to}(r, \varphi \Rightarrow \psi) \end{aligned}$$

$\boxed{\text{cod} : \mathbb{B}^{\to} \to \mathbb{B} \text{ が CCC structure を strict に保つのは定義より明らか}}$

Exercise 1.3.7 (ii)

$\mathbb{B}$がfinite limitを持つ

$\mathbb{B}$がCCC $\Rightarrow$ Sub($\mathbb{B}$)はCCC

terminal object $\begin{pmatrix}1\\ \downarrow !\\ 1\end{pmatrix}$はmono

product $\begin{pmatrix}X\\ \downarrow\varphi\\ I\end{pmatrix}$, $\begin{pmatrix}Y\\ \downarrow\psi\\ J\end{pmatrix}$に対して、

$\begin{pmatrix}X\times Y\\ \downarrow \varphi\times\psi\\ I\times J\end{pmatrix}$はmono

$(\varphi\times\psi)\circ x=(\varphi\times\psi)\circ y$

$\Leftrightarrow \varphi\circ\pi\circ x=\varphi\circ\pi\circ y$

$\wedge\ \psi\circ\pi'\circ x=\psi\circ\pi'\circ y$

$\Leftrightarrow \pi\circ x=\pi\circ x$

$\wedge\ \pi'\circ y=\pi'\circ y$

$\Leftrightarrow x=y$

exponent $\begin{pmatrix}X\\ \downarrow\varphi\\ I\end{pmatrix}$, $\begin{pmatrix}Y\\ \downarrow\psi\\ J\end{pmatrix}$に対して.

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "\varphi\Rightarrow\psi"'] & (X\Rightarrow Y) \arrow[d, "X\Rightarrow\psi"] \\
(I\Rightarrow J) \arrow[r, "\varphi\Rightarrow J"'] & (X\Rightarrow J)
\end{tikzcd}$$

$(X\Rightarrow\psi)\circ f=(X\Rightarrow\psi)\circ g$

$\Leftrightarrow \psi\circ ev\circ(f\times id)$

$=\psi\circ ev\circ(g\times id)$

$\Leftrightarrow ev\circ(f\times id)=ev\circ(g\times id)$

$\Leftrightarrow f=g$

より、$(X\Rightarrow\psi)$はmono

monoのpullbackはmono

なので、$\varphi\Rightarrow\psi$もmono

Exercise 1.3.8　$r: R \to I \times I$ が anti-symmetric であることの定義

$$
\begin{tikzcd}
T \arrow[r, tail, "t"] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, tail, "r"] \\
R \arrow[r, tail, "\langle \pi_2, \pi_1 \rangle \circ r"'] & I \times I
\end{tikzcd}
$$

とおいて

$$
\begin{tikzcd}
T \arrow[rr, dashed] \arrow[dr, tail, "r \circ t"'] & & I \arrow[dl, tail, "\delta_I"] \\
& I \times I &
\end{tikzcd}
$$

Exerise 1.3.9

$$\begin{tikzcd} \mathrm{Sub}(\mathbb{B}) \arrow[d] \\ \mathbb{B} \end{tikzcd} \quad \text{と} \quad \begin{tikzcd} \mathrm{Rel}(\mathbb{B}) \arrow[d] \\ \mathbb{B} \end{tikzcd}$$ については省略

$$\begin{tikzcd} \mathrm{Per}(\mathbb{B}) \arrow[d] \\ \mathbb{B} \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, dashed, "f"] \arrow[d, tail, "r"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, tail, "s"] \\ I \times I \arrow[r, "u \times u"'] & J \times J \end{tikzcd}$$

Sがtransitiveとすると

$$\begin{tikzcd} T \arrow[r, "s_{23}"] \arrow[d, "s_{12}"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "s_1"] \\ S \arrow[r, "s_2"'] & J \end{tikzcd} \qquad \begin{tikzcd} T \arrow[rr, dashed, "g"] \arrow[dr, "t"'] & & S \arrow[dl, "s"] \\ & J \times J & \end{tikzcd}$$

$t = \langle s_1 \circ s_{12}, s_2 \circ s_{23} \rangle$

$$\begin{tikzcd} V \arrow[r, "r_{23}"] \arrow[d, "r_{12}"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, "r_1"] \\ R \arrow[r, "r_2"'] & I \end{tikzcd} \qquad \begin{tikzcd} V \arrow[dr, "v"'] & & R \arrow[dl, "r"] \\ & I \times I & \end{tikzcd}$$

$v = \langle r_1 \circ r_{12}, r_2 \circ r_{23} \rangle$

$$\begin{aligned} (u \times u) \circ v &= \langle u \circ r_1 \circ r_{12}, u \circ r_2 \circ r_{23} \rangle \\ &= \langle s_1 \circ f \circ r_{12}, s_2 \circ f \circ r_{23} \rangle \\ &= \langle s_1 \circ s_{12} \circ h, s_2 \circ s_{23} \circ h \rangle \\ &= t \circ h = s \circ g \circ h \end{aligned}$$

ゆえに r は transitive

$$\begin{tikzcd} V \arrow[r] \arrow[d, "v"'] \arrow[rr, bend left, "h \circ g"] & R \arrow[r, "f"] \arrow[d, "r"] & S \arrow[d, "s"] \\ & I \times I \arrow[r, "u \times u"'] & J \times J \end{tikzcd}$$

$$\begin{aligned} s_2 \circ f \circ r_{12} &= u \circ r_2 \circ r_{12} \\ &= u \circ r_1 \circ r_{23} \\ &= s_1 \circ f \circ r_{23} \end{aligned}$$

$$\begin{tikzcd} V \arrow[rr, "r_{23}"] \arrow[dr, dashed, "h"] \arrow[dd, "r_{12}"'] & & R \arrow[d, "f"] \\ & T \arrow[r, "s_{23}"] \arrow[d, "s_{12}"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "s_1"] \\ R \arrow[r, "f"'] & S \arrow[r, "s_2"'] & J \end{tikzcd}$$

Sがsymmetricとすると

$$\begin{tikzcd} S \arrow[r, dashed, "g"] \arrow[d, "s"'] & S \arrow[d, "s"] \\ J \times J \arrow[r, "\langle \pi', \pi \rangle"'] & J \times J \end{tikzcd}$$

$$\begin{aligned} & (u \times u) \circ \langle \pi', \pi \rangle \circ r \\ &= \langle \pi', \pi \rangle \circ (u \times u) \circ r \\ &= \langle \pi', \pi \rangle \circ s \circ f \\ &= s \circ g \circ f \end{aligned}$$

よって

$$\begin{tikzcd} R \arrow[r] \arrow[d, "r"'] \arrow[rr, bend left, "g \circ f"] & R \arrow[r, "f"] \arrow[d, "r"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, "s"] \\ I \times I \arrow[r, "\langle \pi', \pi \rangle"'] & I \times I \arrow[r, "u \times u"'] & J \times J \end{tikzcd}$$

ゆえに r は symmetric

Exercise 1.3.9 続き

$$\begin{tikzcd} R \arrow[r, dashed, "f"] \arrow[d, "r"'] \arrow[dr, phantom, "\lrcorner"] & S \arrow[d, "s"] \\ I \times I \arrow[r, "u\times u"'] & J \times J \end{tikzcd}$$

sがsymmetric ⇒ rはsymmetric

sがtransitive ⇒ rはtransitive

よって $\begin{pmatrix} \mathrm{Per}(\mathbb{B}) \\ \downarrow \\ \mathbb{B} \end{pmatrix}$ はfibration

sがreflexive

$$\begin{tikzcd} J \arrow[r, dashed, "g"] \arrow[dr, "\delta_J"'] & S \arrow[d, "s"] \\ & J \times J \end{tikzcd} \Rightarrow \begin{tikzcd} I \arrow[r, dashed] \arrow[rr, bend left, "g\circ u"] \arrow[dr, "\delta_I"'] & R \arrow[r] \arrow[d, "r"] & S \arrow[d, "s"] \\ & I \times I \arrow[r, "u\times u"'] & J \times J \end{tikzcd}$$

$$(u\times u)\circ\delta_I = \delta_J \circ u = s\circ g\circ u$$

よって rはreflexive

ゆえに、$\begin{pmatrix} \mathrm{ERel}(\mathbb{B}) \\ \downarrow \\ \mathbb{B} \end{pmatrix}$ はfibration

Exercise 1.3.10

$\mathrm{Rel}'(\mathbb{B})$

objects $R \rightarrowtail I \times J$

morphism from $R \rightarrowtail I \times J$ to $S \rightarrowtail K \times L$ is (u, v) such that

$$\begin{tikzcd} R \arrow[r, dashed] \arrow[d, tail] & S \arrow[d, tail] \\ I \times J \arrow[r, "u \times v"'] & K \times L \end{tikzcd}$$

commutes.

$$\begin{array}{lccc} \mathrm{Rel}'(\mathbb{B}) & R \rightarrowtail I \times J & & (u, v) \\ p \downarrow & \downarrow & & \downarrow \\ \mathbb{B} \times \mathbb{B} & (I, J) & & (u, v) \end{array}$$

p is fibration since monos are stable under pullback.

Exercise 1.3.11

$$\begin{array}{c} \mathbb{E} \\ \downarrow p \\ \mathbb{B} \end{array}$$

p is preordered $\Leftarrow$ p is faithful

$f, g \in \mathbb{E}_I(X, Y)$

$\Rightarrow id_I = p(f) = p(g) \in \mathbb{B}(pX, pY) = \mathbb{B}(I, I)$

$\Rightarrow$ ― pがfaithfulなので ―

$f = g$

よって pは preordered

p is preordered $\Rightarrow$ p is faithful

$f, g \in \mathbb{E}(X, Y) \wedge pf = pg$

$\Rightarrow$ $f = g$ が自明に言えるので

pは faithful

p is preorderd $\wedge$ f is above id_I とする

$f \circ g = f \circ h$

$\Rightarrow p(g) = id_I \circ p(g) = p(f) \circ p(g) = p(f \circ g) = p(f \circ h) = p(h)$

$\Rightarrow g = h$　pはpreorderedで、したがって faithful なので

$\therefore$ fはmono