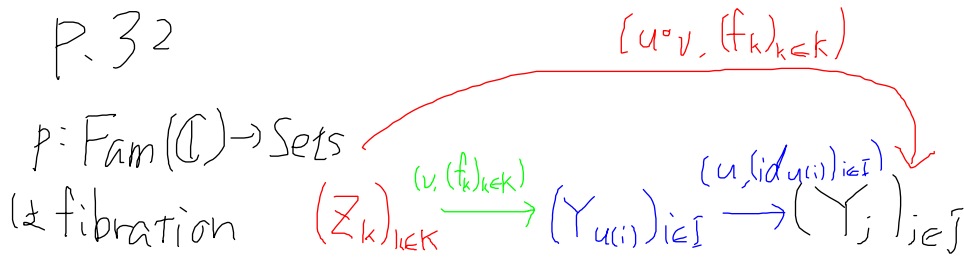


P. 32



$$K \xrightarrow{v} I \xrightarrow{u} J$$

$$\begin{aligned} & (u, (\text{id}_{u(i)})_{i \in I}) \circ (v, (f_k)_{k \in K}) \\ &= (u \circ v, (\text{id}_{u \circ v(k)} \circ f_k)_{k \in K}) \\ &= (u \circ v, (f_k)_{k \in K}) \end{aligned}$$

一意性は自明

p. 37

$$r(X, E) \xrightarrow{f} R \text{ in PER}$$

tracked by e

$$\forall n \in |r(X, E)|. f([n]) = [e \cdot n]$$

let $f^\vee: (X, E) \rightarrow (N/R, \in)$ in ω -sets

$$f^\vee(x) = f([m]) \text{ where } m \in E(x)$$

$$m, m' \in E(x) \Rightarrow [m] = [m'] \text{ なる } \tau$$

well-defined

$$m \in E(x) \Rightarrow f([m]) = [e \cdot m]$$

$$\Rightarrow e \cdot m \in f([m]) = \in(f([m]))$$

なる τ f^\vee は e τ -track される

$$(X, E) \xrightarrow{g} (N/R, \in) \text{ in } \omega\text{-sets}$$

tracked by d

$$\forall x \in X, n \in E(x). d \cdot n \in \in(g(x)) = g(x)$$

$$\text{let } g^\wedge: r(X, E) \rightarrow R \text{ in PER}$$

$$g^\wedge([m]) = g(x) \text{ where } m \in E(x)$$

$$x \sim x' \Rightarrow \exists m. m \in E(x) \wedge m \in E(x') \Rightarrow \exists m. d \cdot m \in g(x) \cap g(x')$$

$$\Rightarrow g(x) = g(x')$$

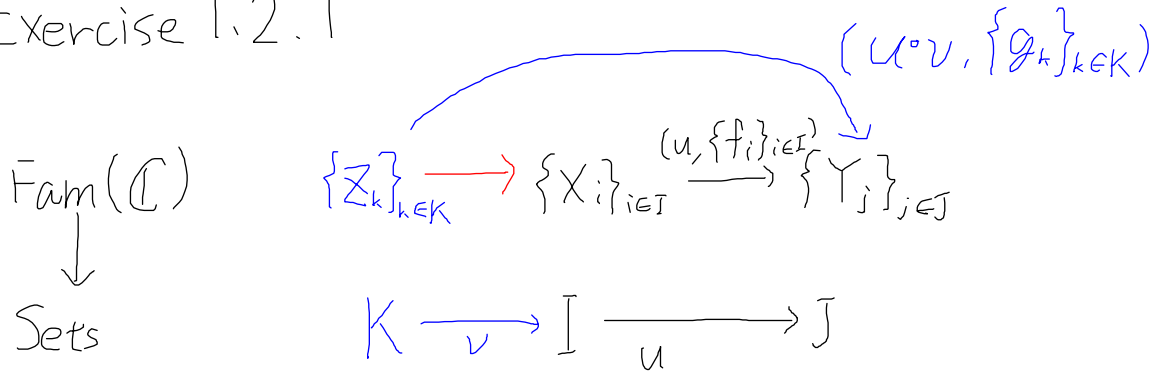
$$x \sim x' \Rightarrow g(x) = g(x') \quad m \in r(X, E) \wedge m' \Rightarrow \exists x, x'. m \in E(x) \wedge m' \in E(x') \wedge x \sim x'$$

\perp, τ g^\wedge is well-defined

$$m' \in E(x') \wedge x \sim x'$$

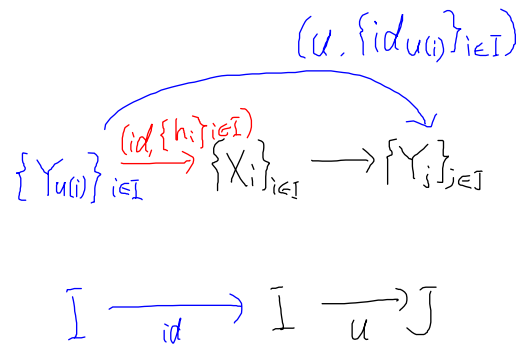
$$\forall m \in |r(X, E)|. g^\wedge([m]) = g(x) = \in(g(x)) = [d \cdot n] \text{ なる } \tau \text{ } d \text{ } \tau \text{-tracked}$$

Exercise 1.2.1

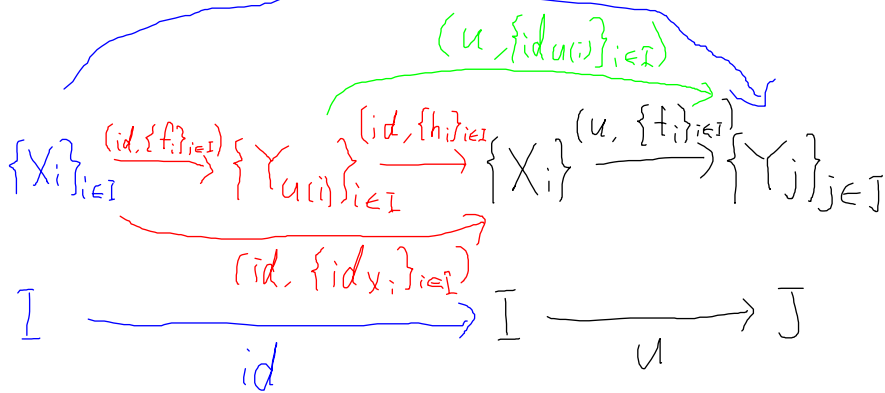


$\forall i \in I. f_i \text{ iso} \Rightarrow (u, \{f_i\}_{i \in I}) \text{ cartesian}$
 $(v, \{f_{v(k)}^{-1} \circ g_k\}_{k \in K})$ が一意な射

$(u, \{f_i\}_{i \in I}) \text{ cartesian} \Rightarrow \forall i \in I. f_i \text{ iso}$



$(u, \{f_i\}_{i \in I}) \therefore f_i \circ h_i = id_{Y_{u(i)}} \text{ for all } i \in I$

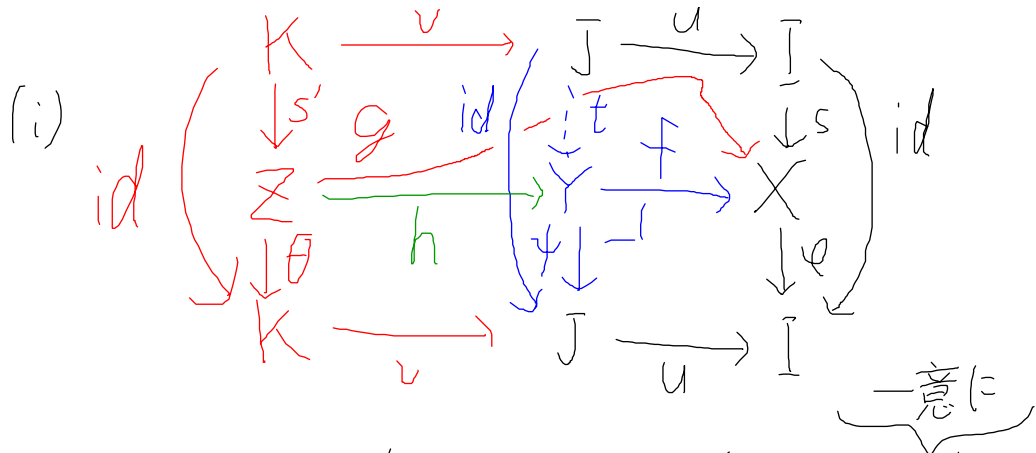


$\therefore h_i \circ f_i = id_{X_i} \text{ for all } i \in I$

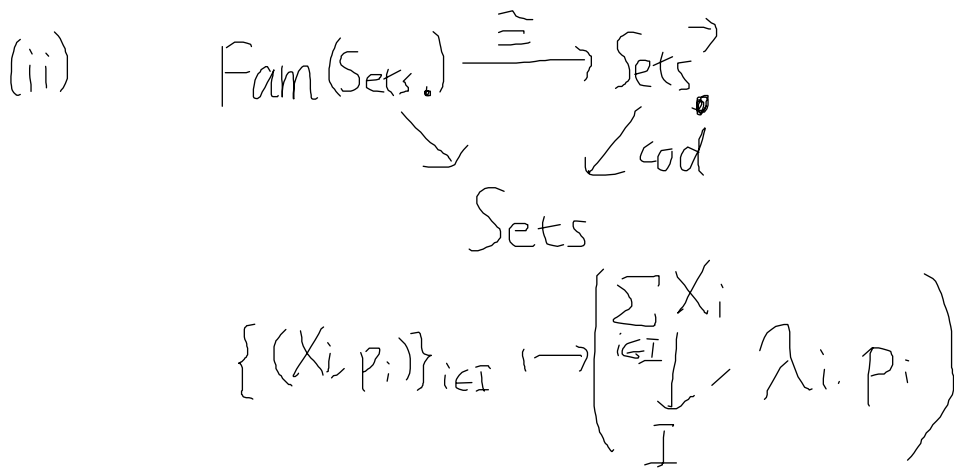
Exercise 1.2.2

trivial

Exercise 1.2.3



Y が"pb"であることを使って、 h を定義
 Y が"pb"であることから $h \circ s' = t \circ v$



Exercise 1.2.4

$P \in R$

$$|R| = \{n \in \mathbb{N} \mid nRn\}$$

$$nRm \Rightarrow nRmRn \Rightarrow nRn \Rightarrow n \in |R|$$

同様 \vdash

$$nRm \Rightarrow m \in m \in |R|$$

$$(n, m) \in R \Rightarrow n \in |R| \wedge m \in |R| \Rightarrow (n, m) \in |R|^2$$

$$\therefore R \subseteq |R|^2$$

Exercise 1.2.5

trivial

Exercise 1.2.6

$$f \in \text{PER}(1, R) \text{ iff } f \in \text{Sets}(\mathbb{N}/1, \mathbb{N}/R) \wedge \exists e. \forall n \in \mathbb{N}. f([n]) = [e \cdot n]$$

$$\text{ここで } \mathbb{N}/1 = \{*\}$$

track が ω - κ は countable

$$\text{すなわち } e = \lambda n. m \text{ where } m \in f^{\omega}(*)$$

とおけば第2の条件は満たせる

$f, g \in (R, S)$ が同じ e によって track されることは、

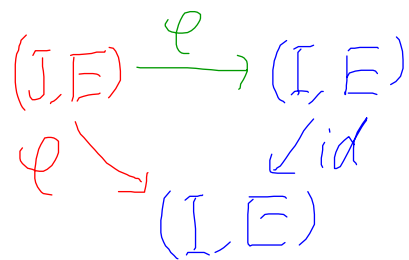
$$\text{よって } \text{PER}(1, R) \cong \text{Sets}(\{*\}, \mathbb{N}/R) \cong \mathbb{N}/R$$

$$\forall n. f([n]) = [e \cdot n] = g([n]) \\ f = g$$

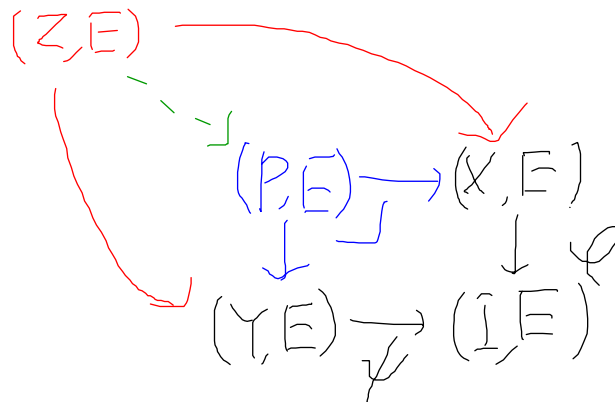
よって $\text{PER}(R, S)$ は countable

Exercise 1.2.7(i) ω -sets / (I, E) ω -CCC

terminal object

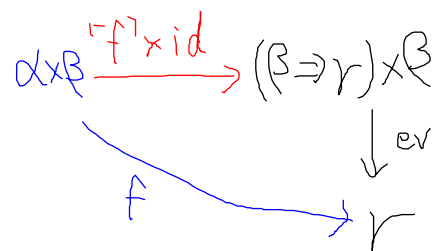


product



exponent

$$\begin{aligned}
 \alpha &: (X, E) \rightarrow (I, E) \\
 \beta &: (Y, E) \rightarrow (I, E) \\
 \gamma &: (Z, E) \rightarrow (I, E)
 \end{aligned}$$



$$\beta \Rightarrow \gamma : (W, E) \rightarrow (I, E)$$

with $W = \sum_{i \in I} (Y_i, E) \rightarrow (Z_i, E)$

$$E_w(i, f) = \{ \langle n, m \rangle \mid n \in E_I(i), m \text{ tracks } f \}$$

$$(\beta \Rightarrow \gamma)(i, f) = i$$

$$f(x) = (i, \lambda y: Y_i. f(x, y)) \text{ where } i \in \alpha(x)$$

$$\text{ev}((i, f), z) = f(z)$$

tracked by $\Delta((i, j), k). j \cdot k$

tracked by $\Delta_n. (e_\alpha \cdot n, \Delta_m. e_f \cdot (n, m))$

Exercise 1.2.7 (ii) PER/I is CCC

terminal object \simeq product \neq id \simeq PB

exponent

$$\alpha: A \rightarrow I$$

$$\beta: B \rightarrow I$$

$$\gamma: C \rightarrow I$$

$$\begin{array}{ccc} \alpha \times \beta & \xrightarrow{\gamma \times \text{id}} & (\beta \Rightarrow \gamma) \times \beta \\ & \searrow f & \downarrow \text{ev} \\ & & \gamma \end{array}$$

$$\begin{aligned} \beta \Rightarrow \gamma: D \rightarrow I \quad D = \{ ((n, m), (n', m')) \mid n \sqsubseteq n', \\ (\beta \Rightarrow \gamma)([(n, m)]) = [n] \quad \forall o, o'. (o \beta o' \wedge \beta([o]) = [n]) \\ \Rightarrow \{ m \cdot o \sqsubseteq m' \cdot o' \wedge \\ \gamma([m \cdot o]) = [n] \} \end{aligned}$$

$$\text{ev}(\{((n, m), o)\}) = [m \cdot o]$$

$$\gamma f([n]) = [(\epsilon_\alpha \cdot n, \wedge m. \epsilon_f \cdot (n, m))]$$

Exercise 1.2.8 $f: \nabla X \rightarrow (N/R, \in)$ in ω -Sets
(\pm constant)

$$\exists e. \forall x \in X. \forall n \in \underbrace{E(b)}_N. e \cdot n \in \underbrace{E(f(x))}_{f(x)}$$

$$\Rightarrow \forall n. e \cdot n \in f(x) \wedge e \cdot n \in f(y)$$

$$\Rightarrow \forall n. e \cdot n \in f(x) \cap f(y)$$

$$\Rightarrow f(x) \cap f(y) \neq \emptyset$$

$$\Rightarrow f(x) = f(y)$$

Exercise 1.2.9 (i)

$PER \simeq \omega\text{-Sets}$

$\eta_{(X,E)} : (X,E) \rightarrow (N/r(X,E), E)$ in $\omega\text{-Sets}$

$\eta_{(X,E)}(x) = [n]_{r(X,E)}$ with $n \in E(x)$

tracked by $\Delta x.x$

E_x が disjoint image を持つとき

$\Rightarrow \eta_{(X,E)}(x) = E(x)$ とき $f : (N/r(X,E), E) \rightarrow (X, E)$
 $f([n]_{r(X,E)}) = x$ with $n \in E(x)$

$\eta_{(X,E)}$ が iso
 のとき

tracked by $\Delta n.n$
 が η の inverse

$E(x) \cap E(y) \neq \emptyset$

$\Rightarrow [n]_{r(X,E)} \supseteq E(x) \cup E(y)$ for some n

$\Rightarrow x = \eta^{-1}([n]) = y$

よって E は disjoint image を持つ

Exercise 1.2.9 (ii)

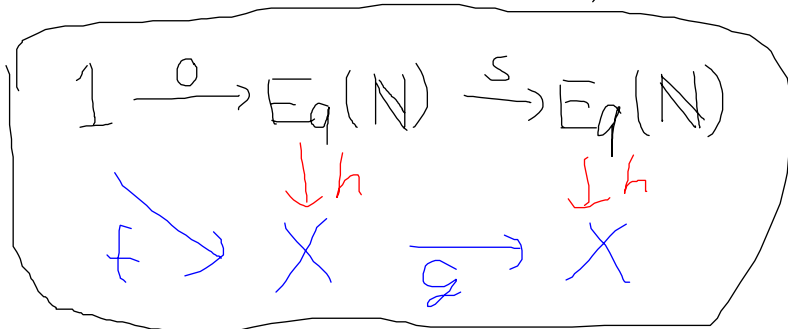
$$r: \omega\text{-Sets} \rightarrow \text{PER}$$

$$(X \rightarrow \text{PN}) \mapsto (X/\sim \rightarrow \text{PN})$$

$$\text{where } \sim = \left\{ (x, y) \mid \begin{array}{l} n \in E(x), m \in E(y) \\ n \sim_{r(x, E)} m \end{array} \right\}$$

Exercise 1.2.10 (i) $Eq(\mathbb{N})$ is NNO

in PER



$$[n]_{Eq(\mathbb{N})} = \{n\}$$

$o(*) = \{0\}$
 tracked by $\lambda x. 0$

$s(\{n\}) = \{n+1\}$
 tracked by $\lambda n. n+1$

$$h(\{0\}) = f(*)$$

$$h(\{n+1\}) = g(h(\{n\}))$$

tracked by $\lambda n. h'(n)$ where $h'(0) = e_f \cdot 0$

$$h'(n+1) = e_g \cdot (h'(n))$$

一意性は明らか

h' is total recursive

ω -Sets の場合も同様にして $N = (\mathbb{N}, \in)$ は NNO

Exercise 1.2.10 (ii)

$f: E_q(\mathbb{N}) \rightarrow E_q(\mathbb{N})$ tracked by $e \mapsto$ total recursive function
 $g(n) = e \cdot n$

total recursive function $f \mapsto g: E_q(\mathbb{N}) \rightarrow E_q(\mathbb{N})$
 $g(\{n\}) = \{f(n)\}$
tracked by
 $\bigwedge n. f(n)$

Exercise 1.2.11 前々半

underlying Set
は Sets 'C' の
tracking されている
ことだけ確認
すれば良い

ω -Sets が finite colimit を持つこと

initial object $0 = (\emptyset, \emptyset)$

!!: $0 \rightarrow X$ is tracked by every $n \in \mathbb{N}$

coproduct $(X, E) + (Y, E) = (X + Y, E)$

with $E(\text{in}_1(x)) = \bar{E}_x(x)$

$E(\text{in}_2(y)) = \bar{E}_y(y)$

$\text{in}_1: (X, E) \rightarrow (X, E) + (Y, E)$ tracked by $\Delta x, 2x$

$\text{in}_2: (Y, E) \rightarrow (X, E) + (Y, E)$ tracked by $\Delta x, 2x+1$

$[f, g]: (X, E) + (Y, E) \rightarrow (Z, E)$ for $f: (X, E) \rightarrow (Z, E), g: (Y, E) \rightarrow (Z, E)$

tracked by Δx . if x is even then $e_f \circ (x/2)$

else $e_g \circ (x/2)$

coequalizer $(X, E) \begin{matrix} \xrightarrow{f} \\ \xrightarrow{g} \end{matrix} (Y, E) \xrightarrow{q} (Q, E)$

with $Q = Y/\sim$

$q' \rightarrow (Z, E)$

\sim is smallest equivalence relation containing $\{f(x), g(x) \mid x \in X\}$

$\bar{E}_Q([y]_{\sim}) = \bigcup_{y' \in [y]_{\sim}} \bar{E}_Y(y')$

$q(y) = [y]$

tracked by $\Delta x, x$

$u([y]) = q'(y)$

tracked by $\Delta n, e_{q'} \circ n$

Note that $y \sim y'$

$\Rightarrow q'(y) = q'(y')$

$\Rightarrow E(q(y)) = E(q(y'))$

Exercise 1.2.11 後半

$$\text{PER} \underset{\text{NAT}}{\overset{r}{\rightleftarrows}} \omega\text{-Sets}$$

$I \xrightarrow{D} \text{PER}$ を finite diagram とする

$$(X, E) = \text{colim } I \xrightarrow{D} \text{PER} \hookrightarrow \omega\text{-Sets} \text{ とする}$$

$$r(X, E) (\cong I \xrightarrow{D} \text{PER} \hookrightarrow \omega\text{-Sets} \rightarrow \text{PER})$$

$$\cong I \xrightarrow{D} \text{PER} \text{ の colimit}$$

left adjoint functor r は colimit を保存

Exercise 1.2.12 前半

$$r: \omega\text{-Sets} \rightarrow \text{PER}$$

r が finite product を保存

$$\begin{aligned} & r(X, E) \times r(Y, E) \\ &= \{ (n, m), (n', m') \mid n \ r(X, E) \ n', \ m \ r(Y, E) \ m' \} \\ &= \{ (n, m), (n', m') \mid \exists x, x' \in X. n \in E(x) \wedge n' \in E(x') \wedge n \sim n', \\ & \quad \exists y, y' \in Y. m \in E(y) \wedge m' \in E(y') \wedge m \sim m' \} \end{aligned}$$

$$\begin{aligned} & r((X, E) \times (Y, E)) \\ &= \{ (n, n') \mid \exists (x, y), (x', y') \in X \times Y. n \in E(x, y) \wedge n' \in E(x', y') \wedge \\ & \quad n \sim n' \} \\ &= \{ ((n, m), (n', m')) \mid \exists x, x' \in X. \exists y, y' \in Y. (n, m) \in E(x, y) \wedge \\ & \quad (n', m') \in E(x', y') \wedge (n, m) \sim (n', m') \} \\ &= \{ ((n, m), (n', m')) \mid \exists x, x' \in X. n \in E(x) \wedge n' \in E(x') \wedge n \sim n', \\ & \quad \exists y, y' \in Y. m \in E(y) \wedge m' \in E(y') \wedge m \sim m' \} \\ &= r(X, E) \times r(Y, E) \end{aligned}$$

$$\begin{aligned} r(\{*\}, E) &= \{ (n, n') \mid n \in E(*), n' \in E(*), n \sim n' \} \\ &= E(*)^2 \end{aligned}$$

Exercise 1.2.12 後半

$$\emptyset \longrightarrow 2 \begin{array}{c} \xrightarrow{\text{id}} \\ \xrightarrow{\neg} \end{array} 2 \text{ in Sets } \text{ is equalizer}$$

$$\nabla \emptyset \longrightarrow \nabla 2 \begin{array}{c} \xrightarrow{\nabla \text{id}} \\ \xrightarrow{\nabla \neg} \end{array} \nabla 2 \text{ in } \omega\text{-Sets } \text{ is equalizer}$$

$$r\nabla \emptyset \longrightarrow r\nabla 2 \begin{array}{c} \xrightarrow{r\nabla \text{id}} \\ \xrightarrow{r\nabla \neg} \end{array} r\nabla 2 \text{ in PER}$$

$$r\nabla 2 = \mathbb{N} \times \mathbb{N}$$

$$r\nabla \text{id}(\mathbb{N}) = \mathbb{N} = r\nabla \neg \quad \text{ただし } (=)$$

$$r\nabla \emptyset = \emptyset \neq r\nabla 2$$

\therefore equalizer τ (ただし)

$\therefore r$ は equalizer を 保存しない

Exercise 1.2.13 (i)

$\text{Fam}(-) : \text{Cats} \rightarrow \text{Cats}$ が 2 functor であること

定義

$$F : \mathbb{C} \rightarrow \mathbb{D} \text{ に対して}$$

$$\text{Fam}(F) : \text{Fam}(\mathbb{C}) \rightarrow \text{Fam}(\mathbb{D})$$

$$\text{Fam}(F)(\{X_i\}_{i \in I}) = \{FX_i\}_{i \in I}$$

$$\text{Fam}(F)((u, \{f_i\}_{i \in I}) : \{X_i\}_{i \in I} \rightarrow \{Y_j\}_{j \in J})$$

$$= (u, \{Ff_i\}_{i \in I}) : \{FX_i\}_{i \in I} \rightarrow \{FY_j\}_{j \in J}$$

条件

$$\text{Fam}(\text{Id}_{\mathbb{C}}) = \text{Id}_{\text{Fam}(\mathbb{C})}$$

$$\text{Fam}(G \circ F) = \text{Fam}(G) \circ \text{Fam}(F)$$

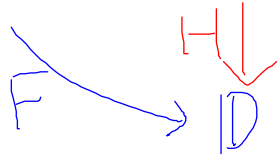
$$\text{Fam}(F)(\text{id}_{\{X_i\}_{i \in I}}) = \text{id}_{\text{Fam}(F)(\{X_i\}_{i \in I})}$$

$$\text{Fam}(F)((v, \{g_j\}_{j \in J}) \circ (u, \{f_i\}_{i \in I})) = \text{Fam}(F)(v, \{g_j\}_{j \in J})$$

$$\circ \text{Fam}(F)(u, \{f_i\}_{i \in I})$$

Exercise 1.2.13 (ii)

$$\mathbb{C} \xrightarrow{\text{unit}} \text{Fam}(\mathbb{C})$$



\mathbb{D} with set-indexed coproduct

$$\bigsqcup_{i \in I} \{X_{ij}\}_{j \in J_i} \xrightarrow{g} \{Y_k\}_{k \in K}$$

$\uparrow \text{in}_i$

$$\{X_{i,j}\}_{j \in J_i} \xrightarrow{(u_i, \{f_{i,j}\}_{j \in J_i})}$$

$$\bigsqcup_{i \in I} \{X_j\}_{j \in J_i} \triangleq \{X_{i,j}\}_{(i,j) \in \Sigma_{i \in I, J_i}}$$

$$\text{in}_i \triangleq (u_i, \{ \text{in}_{i,j} \}_{j \in J_i})$$

with $u_i(j) = (i,j)$

$$\text{in}_{i,j}(x) = x$$

$$g \triangleq (w, \{g_{i,j}\}_{(i,j) \in \Sigma_{i \in I, J_i}})$$

with $w(i,j) = u_i(j)$

$$g_{i,j}(x) = f_{i,j}(x)$$

$$\text{unit}(X) = \{X\}_{i \in 1}$$

$$\text{unit}(f) = (\text{id}_1, \{f\}_{i \in 1})$$

$$H(\{X_i\}_{i \in I}) = \bigsqcup_{i \in I} FX_i$$

$$H((u, \{f_i\}_{i \in I}) : \{X_i\}_{i \in I} \rightarrow \{Y_j\}_{j \in J})$$

$$= [\text{in}_{u(i)} \circ Ff_i]_{i \in I} : \bigsqcup_{i \in I} X_i \rightarrow \bigsqcup_{j \in J} Y_j$$

$$H \text{unit}(X) = \bigsqcup_{i \in 1} FX \cong FX$$

$$FX \xrightarrow{Ff} FY$$

$\cong \circ \cong$

Hの一意性は
明らか

$$H \text{unit}(f) = [\text{in}_i \circ Ff]_{i \in 1} : \bigsqcup_{i \in 1} FX \rightarrow \bigsqcup_{i \in 1} FY$$

Exercise 1.2.13 (iii)

$$\text{unit} : \mathbb{C} \rightarrow \text{Fam}(\mathbb{C})$$

$F \dashv \text{unit}$

$$\begin{aligned} \Leftrightarrow \mathbb{C}(FX, Y) &\cong \text{Fam}(\mathbb{C})(\{X_i\}_{i \in I}, \text{unit } Y) \\ &\cong \prod_{i \in I} \mathbb{C}(X_i, Y) \end{aligned}$$