P.32

$p : \mathrm{Fam}(\mathbb{C}) \to \mathrm{Sets}$ は fibration

$$
\begin{tikzcd}
(Z_k)_{k\in K} \arrow[r, "{(v, (f_k)_{k\in K})}"] \arrow[rr, bend left, "{(u\circ v, (f_k)_{k\in K})}"] & (Y_{u(i)})_{i\in I} \arrow[r, "{(u, (id_{u(i)})_{i\in I})}"] & (Y_j)_{j\in J}
\end{tikzcd}
$$

$$K \xrightarrow[v]{} I \xrightarrow[u]{} J$$

$$
\begin{aligned}
&(u, (id_{u(i)})_{i\in I}) \circ (v, (f_k)_{k\in K}) \\
&= (u\circ v, (id_{u(v(k))} \circ f_k)_{k\in K}) \\
&= (u\circ v, (f_k)_{k\in K})
\end{aligned}
$$

一意性は自明

p. 37

$r(X,E) \xrightarrow{f} R$ in PER

tracked by e

$\forall n \in |r(X,E)|.\ f([n]) = [e \cdot n]$

let $f^{\vee} : (X,E) \to (\mathbb{N}/R, \in)$ in $\omega$-Sets

$f^{\vee}(x) = f([m])$ where $m \in E(x)$

$m, m' \in E(x) \Rightarrow [m] = [m']$ なので well-defined

$m \in E(x) \Rightarrow f([m]) = [e \cdot m]$

$\Rightarrow e \cdot m \in f([m]) = \in(f([m]))$

なので $f^{\vee}$ は $e$ で track される

$(X,E) \xrightarrow{g} (\mathbb{N}/R, \in)$ in $\omega$-Sets

tracked by d

$\forall x \in X, n \in E(x).\ d \cdot n \in \in(g(x)) = g(x)$

let $g^{\wedge} : r(X,E) \to R$ in PER

$g^{\wedge}([m]) = g(x)$ where $m \in E(x)$

$x \smile x' \Rightarrow \exists m.\ m \in E(x) \wedge m \in E(x') \Rightarrow^{\exists m} d \cdot m \in g(x) \cap g(x')$

$\Rightarrow g(x) = g(x')$

$x \sim x' \Rightarrow g(x) = g(x')$ $\quad m\ r(X,E)\ m' \Rightarrow \exists x, x'.\ m \in E(x) \wedge m' \in E(x') \wedge x \sim x'$

よって $g^{\wedge}$ は well-defined

$\forall m \in |r(X,E)|.\ g^{\wedge}([m]) = g(x) = \in(g(x)) = [d \cdot n]$ なので $d$ で tracked

# Exercise 1.2.1

$$\begin{tikzcd} \mathrm{Fam}(\mathbb{C}) \arrow[d] \\ \mathrm{Sets} \end{tikzcd}
\qquad
\begin{tikzcd} \{Z_k\}_{k\in K} \arrow[r] \arrow[rr, bend left, "(u\circ v,\ \{g_k\}_{k\in K})"] & \{X_i\}_{i\in I} \arrow[r, "(u,\{f_i\}_{i\in I})"] & \{Y_j\}_{j\in J} \\ K \arrow[r, "v"'] & I \arrow[r, "u"'] & J \end{tikzcd}$$

$\forall i\in I.\ f_i$ iso $\Rightarrow$ $(u,\{f_i\}_{i\in I})$ cartesian

$(v, \{f_{v(k)}^{-1}\circ g_k\}_{k\in K})$ が一意な射

$(u,\{f_i\}_{i\in I})$ cartesian $\Rightarrow$ $\forall i\in I.\ f_i$ iso

$$\begin{tikzcd} \{Y_{u(i)}\}_{i\in I} \arrow[r, "(id,\{h_i\}_{i\in I})"] \arrow[rr, bend left, "(u,\{id_{u(i)}\}_{i\in I})"] & \{X_i\}_{i\in I} \arrow[r] & \{Y_j\}_{j\in J} \\ I \arrow[r, "id"'] & I \arrow[r, "u"'] & J \end{tikzcd}$$

$$\begin{tikzcd} \{X_i\}_{i\in I} \arrow[r, "(id,\{f_i\}_{i\in I})"] \arrow[rrr, bend left=40, "(u,\{f_i\}_{i\in I})"] \arrow[rr, bend right, "(id,\{id_{X_i}\}_{i\in I})"'] & \{Y_{u(i)}\}_{i\in I} \arrow[r, "(id,\{h_i\}_{i\in I})"] \arrow[rr, bend left, "(u,\{id_{u(i)}\}_{i\in I})"] & \{X_i\} \arrow[r, "(u,\{f_i\}_{i\in I})"] & \{Y_j\}_{j\in J} \\ I \arrow[rr, "id"'] & & I \arrow[r, "u"'] & J \end{tikzcd}$$

$\therefore\ f_i\circ h_i = id_{Y_{u(i)}}$ for all $i\in I$

$\therefore\ h_i\circ f_i = id_{X_i}$ for all $i\in I$

Exercise 1.2.2

trivial

# Exercise 1.2.3

(i)

$$\begin{tikzcd}
K \arrow[r, "v"] \arrow[d, "s'"] \arrow[dd, bend right=60, "\mathrm{id}"'] & J \arrow[r, "u"] \arrow[d, dashed, "t"] \arrow[dd, bend right=60, "\mathrm{id}"'] & I \arrow[d, "s"] \arrow[dd, bend left=60, "\mathrm{id}"] \\
Z \arrow[r, "h"] \arrow[d, "\theta"] \arrow[urr, bend left, "g"] & Y \arrow[r, "f"] \arrow[d, "\psi"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\varphi"] \\
K \arrow[r, "v"'] & J \arrow[r, "u"'] & I
\end{tikzcd}$$

Yがpbであることを使って、$\overbrace{h\text{を定義}}^{\text{一意に}}$

Yがpbであることから $h\circ s' = t\circ v$

(ii)

$$\begin{tikzcd}
\mathrm{Fam}(\mathrm{Sets}_\bullet) \arrow[rr, "\cong"] \arrow[dr] & & \mathrm{Sets}^{\to}_{\bullet} \arrow[dl, "\mathrm{cod}"] \\
& \mathrm{Sets} &
\end{tikzcd}$$

$$\{(X_i, p_i)\}_{i\in I} \longmapsto \left( \begin{array}{c} \sum_{i\in I} X_i \\ \downarrow \\ I \end{array}, \ \lambda i.\, p_i \right)$$

# Exercise 1.2.4

PER $R$ $\qquad\qquad$ $|R| = \{n \in \mathbb{N} \mid nRn\}$

$nRm \Rightarrow nRmRn \Rightarrow nRn \Rightarrow n \in |R|$

同様に

$nRm \Rightarrow \cancel{m \in}\ m \in |R|$

$(n, m) \in R \Rightarrow n \in |R| \land m \in |R| \Rightarrow (n, m) \in |R|^2$

$\therefore\ R \subseteq |R|^2$

# Exercise 1.2.5

trivial

# Exercise 1.2.6

$f \in PER(1, R)$ iff $f \in Sets(\mathbb{N}/1, \mathbb{N}/R) \land$
$\exists e. \forall n \in |1|. \ f([n]) = [e \cdot n]$

ここで $\mathbb{N}/1 = \{*\}$

また $e = \Lambda n. m$ where
$m \in f(*)$
とおけば第2の条件は
満たせる

よって $PER(1, R) \cong Sets(\{*\}, \mathbb{N}/R)$
$\cong \mathbb{N}/R$

trackするコードは
countable

$f, g \in (R, S)$ が
同じeによって
trackされるとす
ると、
$\forall n. f([n]) = [e \cdot n] = g([n])$
$f = g$

よって $PER(R, S)$ はcountable

# Exercise 1.2.7(i) ω-sets/(I,E) がCCC

terminal object

$$\begin{tikzcd}
(J,E) \arrow[r, "\varphi"] \arrow[dr, "\varphi"'] & (I,E) \arrow[d, "id"] \\
 & (I,E)
\end{tikzcd}$$

product

$$\begin{tikzcd}
(Z,E) \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
 & (P,E) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (X,E) \arrow[d, "\varphi"] \\
 & (Y,E) \arrow[r, "\psi"'] & (I,E)
\end{tikzcd}$$

exponent

$$\begin{tikzcd}
\alpha\times\beta \arrow[r, "\ulcorner f\urcorner\times id"] \arrow[dr, "f"'] & (\beta\Rightarrow\gamma)\times\beta \arrow[d, "ev"] \\
 & \gamma
\end{tikzcd}$$

$\alpha : (X,E) \to (I,E)$
$\beta : (Y,E) \to (I,E)$
$\gamma : (Z,E) \to (I,E)$

$\beta\Rightarrow\gamma : (W,E) \to (I,E)$

with $W = \sum_{i\in I} (Y_i,E) \to (Z_i,E)$

$E_W(i,f) = \{\langle n,m\rangle \mid n\in E_I(i),\ m \text{ tracks } f\}$

$(\beta\Rightarrow\gamma)(i,f) = i$

$ev((i,f),y) = f(y)$

tracked by $\Lambda((i,j),k).\,j\cdot k$

$\ulcorner f\urcorner(x) = (i, \lambda y{:}Y_i.\, f(x,y))$ where $i\in\alpha(x)$

tracked by $\Lambda n.\,(e_\alpha\cdot n,\ \Lambda m.\, e_f\cdot(n,m))$

# Exercise 1.2.7 (ii)  PER/I は CCC

terminal object と product は id と PB

exponent

$\alpha : A \to I$

$\beta : B \to I$

$\gamma : C \to I$

$$\begin{tikzcd} \alpha \times \beta \arrow[r, "\ulcorner f \urcorner \times id"] \arrow[dr, "f"'] & (\beta \Rightarrow \gamma) \times \beta \arrow[d, "ev"] \\ & \gamma \end{tikzcd}$$

$\beta \Rightarrow \gamma : D \to I$

$(\beta \Rightarrow \gamma)([(n,m)]) = [n]$

$$D = \{ ((n,m),(n',m')) \mid n I n', \forall o, o'. (o B o' \wedge \beta([o]) = [n] \Rightarrow (m \cdot o \; C \; m' \cdot o' \wedge \gamma([m \cdot o]) = [n])) \}$$

$$ev([((n,m),o)]) = [m \cdot o]$$

$$\ulcorner f \urcorner([n]) = [(e_\alpha \cdot n, \Lambda m. e_f \cdot (n,m))]$$

Exercise 1.2.8 $f : \nabla X \to (\mathbb{N}/R, \in)$ in $\omega$-Sets

is constant

$$\exists e.\ \forall x \in X.\ \forall n \in \underbrace{E(x)}_{= \mathbb{N}}.\ e \cdot n \in \underbrace{E(f(x))}_{= f(x)}$$

$$\Rightarrow \forall n.\ e \cdot n \in f(x) \wedge e \cdot n \in f(y)$$

$$\Rightarrow \forall n.\ e \cdot n \in f(x) \cap f(y)$$

$$\Rightarrow f(x) \cap f(y) \neq \emptyset$$

$$\Rightarrow f(x) = f(y)$$

# Exercise 1.2.9 (i)

$\mathrm{PER} \leftrightarrows \omega\text{-Sets}$

$\eta_{(X,E)} : (X,E) \to (\mathbb{N}/r(X,E), \in)$ in $\omega$-Sets

$\eta_{(X,E)}(x) = [n]_{r(X,E)}$ with $n \in E(x)$

tracked by $\Lambda x.x$

$E_x$がdisjoint imageを持つとき

$\Rightarrow \eta_{(X,E)}(x) = E(x)$で $f : (\mathbb{N}/r(X,Y), \in) \to (X,E)$

$f([n]_{r(X,E)}) = x$ with $n \in E(x)$

tracked by $\Lambda n.n$

が$\eta$のinverse

$\eta_{(X,E)}$がisoのとき

$E(x) \cap E(y) \neq \emptyset$

$\Rightarrow \quad [n]_{r(X,E)} \supseteq E(x) \cup E(y)$ for some $n$

$\Rightarrow \quad x = \eta^{-1}([n]) = y$

よってEはdisjoint imageを持つ

# Exercise 1.2.9 (ii)

$r : \omega\text{-Sets} \longrightarrow PER$

$(X \to P\mathbb{N}) \mapsto (X/\sim \ \to P\mathbb{N})$

where $\sim \ = \{ (x, y) \mid n \in E(x), m \in E(y), \; n \sim_{r(X,E)} m \}$

# Exercise 1.2.10 (i)　Eq(N)がNNO in PER

$$\begin{tikzcd}
1 \arrow[r, "0"] \arrow[dr, "f"'] & Eq(\mathbb{N}) \arrow[r, "s"] \arrow[d, "h"] & Eq(\mathbb{N}) \arrow[d, "h"] \\
 & X \arrow[r, "g"'] & X
\end{tikzcd}$$

$$[n]_{Eq(\mathbb{N})} = \{n\}$$

$0(*) = \{0\}$
tracked by $\Lambda x.\, 0$

$s(\{n\}) = \{n+1\}$
tracked by $\Lambda n.\, n+1$

$h(\{0\}) = f(*)$

$h(\{n+1\}) = g(h(\{n\}))$

tracked by $\Lambda n.\, h'(n)$ where $h'(0) = e_f \cdot 0$

$h'(n+1) = e_g \cdot (h'(n))$

一意性は明らか

$h'$は total recursive

$\omega$-Setsの場合も同様にして $N = (\mathbb{N}, \in)$ は NNO

# Exercise 1.2.10 (ii)

$f : Eq(\mathbb{N}) \to Eq(\mathbb{N}) \longmapsto$ total recursive function
tracked by $e$ $\qquad g(n) = e \cdot n$

total recursive function $f \longmapsto g : Eq(\mathbb{N}) \to Eq(\mathbb{N})$
$g(\{n\}) = \{f(n)\}$
tracked by
$\Lambda n. f(n)$

# Exercise 1.2.11 前半

underlying set は Sets と同じ。
tracking されていることだけ確認すれば良い

$\omega$-Sets が finite colimit を持つこと

· initial object $0=(\emptyset,\emptyset)$

$!:0\to X$ is tracked by every $n\in\mathbb{N}$

· coproduct $(X,E)+(Y,E)=(X+Y,E)$

with $E(\mathrm{in}_1(x))=E_X(x)$

$E(\mathrm{in}_2(y))=E_Y(y)$

$\mathrm{in}_1:(X,E)\to(X,E)+(Y,E)$ tracked by $\Lambda x.\,2x$

$\mathrm{in}_2:(Y,E)\to(X,E)+(Y,E)$ tracked by $\Lambda x.\,2x+1$

$[f,g]:(X,E)+(Y,E)\to(Z,E)$ for $f:(X,E)\to(Z,E)$, $g:(Y,E)\to(Z,E)$

tracked by $\Lambda x.$ if $x$ is even then $e_f\bullet(x/2)$

else $e_g\bullet\lfloor x/2\rfloor$

coequalizer
$$\begin{tikzcd}
(X,E) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Y,E) \arrow[r, "q"] \arrow[dr, "q'"'] & (Q,E) \arrow[d, "u"] \\
 & & (Z,E)
\end{tikzcd}$$

with $Q=Y/\sim$

$\sim$ is smallest equivalence relation containing $\{(f(x),g(x))\mid x\in X\}$

$E_Q([y]_\sim)=\bigcup_{y'\in[y]}E_Y(y')$

$q(y)=[y]$

tracked by $\Lambda x.x$

$u([y])=q'(y)$

tracked by $\Lambda n.\,e_{q'}\bullet n$

Note that $y\sim y'$
$\Rightarrow q'(y)=q'(y')$
$\Rightarrow E(q'(y))=E(q'(y'))$

# Exercise 1.2.11 後半

$$\mathrm{PER} \underset{\mathbb{N}/(-)}{\overset{r}{\leftrightarrows}} \omega\text{-Sets}$$

$I \xrightarrow{D} \mathrm{PER}$ を finite diagram とする

$(X, E) = \mathrm{colim}\ I \xrightarrow{D} \mathrm{PER} \hookrightarrow \omega\text{-Sets}$ とすると

$r(X, E)$ は $I \xrightarrow{D} \mathrm{PER} \hookrightarrow \omega\text{-Sets} \to \mathrm{PER}$

$\cong I \xrightarrow{D} \mathrm{PER}$ の colimit

left adjoint functor $r$ は colimit を保存

# Exercise 1.2.12 前半

$r : \omega\text{-Sets} \to PER$

$r$がfinite productを保存

$$
\begin{aligned}
& r(X, E) \times r(Y, E) \\
&= \{ ((n, m), (n', m')) \mid n \; r(X, E) \; n', \; m \; r(Y, E) \; m' \} \\
&= \{ ((n, m), (n', m')) \mid \exists x, x' \in X.\ n \in E(x) \wedge n' \in E(x') \wedge n \sim n', \\
&\qquad \exists y, y' \in Y.\ m \in E(y) \wedge m' \in E(y') \wedge m \sim m' \}
\end{aligned}
$$

$$
\begin{aligned}
& r((X, E) \times (Y, E)) \\
&= \{ (n, n') \mid \exists (x, y), (x', y') \in X \times Y.\ n \in E(x, y) \wedge n' \in E(x', y') \wedge \\
&\qquad n \sim n' \} \\
&= \{ ((n, m), (n', m')) \mid \exists x, x' \in X.\ \exists y, y' \in Y.\ (n, m) \in E(x, y) \wedge \\
&\qquad (n', m') \in E(x', y') \wedge (n, m) \sim (n', m') \} \\
&= \{ ((n, m), (n', m')) \mid \exists x, x'.\ n \in E(x) \wedge n' \in E(x') \wedge n \sim n', \\
&\qquad \exists y, y'.\ m \in E(y) \wedge m' \in E(y') \wedge m \sim m' \} \\
&= r(X, E) \times r(Y, E)
\end{aligned}
$$

$$
\begin{aligned}
r(\{*\}, E) &= \{ (n, n') \mid n \in E(*), n' \in E(*), n \sim n' \} \\
&= E(*)^2
\end{aligned}
$$

# Exercise 1.2.12 後半

$$\emptyset \longrightarrow 2 \underset{\neg}{\overset{id}{\rightrightarrows}} 2$$ in Sets は equalizer

$$\nabla\emptyset \longrightarrow \nabla 2 \underset{\nabla\neg}{\overset{\nabla id}{\rightrightarrows}} \nabla 2$$ in $\omega$-Sets は equalizer

$$r\nabla\emptyset \longrightarrow r\nabla 2 \underset{r\nabla\neg}{\overset{r\nabla id}{\rightrightarrows}} r\nabla 2$$ in PER

$r\nabla 2 = \mathbb{N}\times\mathbb{N}$

$r\nabla id(\mathbb{N}) = \mathbb{N} = r\nabla\neg$ なのに

$r\nabla\emptyset = \emptyset \neq r\nabla 2$

$\therefore$ equalizer ではない

$\therefore$ r は equalizer を保存しない

# Exercise 1.2.13 (i)

$\mathrm{Fam}(-): \mathcal{C}ats \to \mathcal{C}ats$ が 2 functorであること

定義　$F : \mathbb{C} \to \mathbb{D}$ に対して

$\mathrm{Fam}(F) : \mathrm{Fam}(\mathbb{C}) \to \mathrm{Fam}(\mathbb{D})$

$\mathrm{Fam}(F)(\{X_i\}_{i\in I}) = \{FX_i\}_{i\in I}$

$\mathrm{Fam}(F)((u, \{f_i\}_{i\in I}) : \{X_i\}_{i\in I} \to \{Y_j\}_{j\in I})$

$= (u, \{Ff_i\}_{i\in I}) : \{FX_i\}_{i\in I} \to \{FY_j\}_{j\in J}$

条件　$\mathrm{Fam}(\mathrm{Id}_{\mathbb{C}}) = \mathrm{Id}_{\mathrm{Fam}(\mathbb{C})}$

$\mathrm{Fam}(G\circ F) = \mathrm{Fam}(G)\circ \mathrm{Fam}(F)$

$\mathrm{Fam}(F)(id_{\{X_i\}_{i\in I}}) = id_{\mathrm{Fam}(F)(\{X_i\}_{i\in I})}$

$\mathrm{Fam}(F)((v, \{g_j\}_{j\in J})\circ(u, \{f_i\}_{i\in I})) = \mathrm{Fam}(F)(v, \{g_j\}_{j\in J})$

$\circ \mathrm{Fam}(F)(u \{f_i\}_{i\in I})$

# Exercise 1.2.13 (ii)

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, "\mathrm{unit}"] \arrow[dr, "F"'] & \mathrm{Fam}(\mathbb{C}) \arrow[d, "H"] \\
 & \mathbb{D}
\end{tikzcd}
$$

$\mathbb{D}$ with set-indexed coproduct

$$
\begin{tikzcd}
\coprod_{i\in I}\{X_{i,j}\}_{j\in J_i} \arrow[r, "g"] & \{Y_k\}_{k\in K} \\
\{X_{i,j}\}_{j\in J_i} \arrow[u, "in_i"] \arrow[ur, "(u_i, \{f_{i,j}\}_{j\in J_i})"'] &
\end{tikzcd}
$$

$\coprod_{i\in I}\{X_j\}_{j\in J_i} \triangleq \{X_{i,j}\}_{(i,j)\in\Sigma_{i\in I}.J_i}$

$in_i \triangleq (v_i, \{in_{i,j}\}_{j\in J})$

with $v_i(j) = (i,j)$

$in_{i,j}(x) = x$

$g \triangleq (w, \{g_{i,j}\}_{(i,j)\in\Sigma_{i\in I}.J_i})$

with $w(i,j) = u_i(j)$

$g_{i,j}(x) = f_{i,j}(x)$

$\mathrm{unit}(X) = \{X\}_{i\in 1}$

$\mathrm{unit}(f) = (id_1, \{f\}_{i\in 1})$

$H(\{X_i\}_{i\in I}) = \coprod_{i\in I} FX_i$

$H((u, \{f_i\}_{i\in I}) : \{X_i\}_{i\in I} \to \{Y_j\}_{j\in J})$

$= [in_{u(i)} \circ Ff_i]_{i\in I} : \coprod_{i\in I} X_i \to \coprod_{j\in J} Y_j$

$H\,\mathrm{unit}(X) = \coprod_{i\in 1} FX \cong FX$

$$
\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[d, phantom, "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & FY \arrow[d, phantom, "\cong"] \\
{} & {}
\end{tikzcd}
$$

Hの一意性は明らか

$H\,\mathrm{unit}(f) = [in_i \circ Ff]_{i\in 1} : \coprod_{i\in 1} FX \to \coprod_{i\in 1} FY$

# Exercise 1.2.13 (iii)

$$\mathrm{unit} : \mathbb{C} \to \mathrm{Fam}(\mathbb{C})$$

$$F \dashv \mathrm{unit}$$

$$\Leftrightarrow \mathbb{C}(FX, Y) \cong \mathrm{Fam}(\mathbb{C})(\{X_i\}_{i \in I}, \mathrm{unit}\, Y)$$

$$\cong \prod_{i \in I} \mathbb{C}(X_i, Y)$$