Theorem 1 *If*

$$\mathsf{p} \cdot \mathsf{q} \quad = \quad \mathsf{q} \cdot \mathsf{r} \tag{1}$$

then

$$\mathsf{p}^* \cdot \mathsf{q} = \mathsf{q} \cdot \mathsf{r}^*$$

By antisym, it suffices to show that

$$\mathsf{p}^* \cdot \mathsf{q} \quad \leq \quad \mathsf{q} \cdot \mathsf{r}^* \tag{2}$$
$$\mathsf{q} \cdot \mathsf{r}^* \quad \leq \quad \mathsf{p}^* \cdot \mathsf{q} \tag{3}$$

Consider (2). By *R, it suffices to show that

$$\mathsf{p} \cdot \mathsf{q} \cdot \mathsf{r}^* + \mathsf{q} \quad \leq \quad \mathsf{q} \cdot \mathsf{r}^* \tag{4}$$

Consider (4). By unwindL, we know that

$$\mathsf{q} \cdot \mathsf{r}^* \quad = \quad \mathsf{q} \cdot (1 + \mathsf{r} \cdot \mathsf{r}^*)$$

By distrL, we know that

$$\mathsf{q} \cdot (1 + \mathsf{r} \cdot \mathsf{r}^*) \quad = \quad \mathsf{q} \cdot 1 + \mathsf{q} \cdot \mathsf{r} \cdot \mathsf{r}^*$$

By id.R, we know that

$$\mathsf{q} \cdot 1 + \mathsf{q} \cdot \mathsf{r} \cdot \mathsf{r}^* \quad = \quad \mathsf{q} + \mathsf{q} \cdot \mathsf{r} \cdot \mathsf{r}^*$$

By commut+, we know that

$$\mathsf{q} + \mathsf{q} \cdot \mathsf{r} \cdot \mathsf{r}^* \quad = \quad \mathsf{q} \cdot \mathsf{r} \cdot \mathsf{r}^* + \mathsf{q}$$

By mono+R, it suffices to show that

$$\mathsf{p} \cdot \mathsf{q} \cdot \mathsf{r}^* \quad \leq \quad \mathsf{q} \cdot \mathsf{r} \cdot \mathsf{r}^* \tag{5}$$

Consider (5). By mono.R, it suffices to show that

$$\mathsf{p} \cdot \mathsf{q} \quad \leq \quad \mathsf{q} \cdot \mathsf{r} \tag{6}$$

Consider (6). By =¡, it suffices to show that

$$\mathsf{p} \cdot \mathsf{q} \quad = \quad \mathsf{q} \cdot \mathsf{r} \tag{7}$$

Consider (7). By (1), we have what we need. Consider (3). By *L, it suffices to show that

$$\mathsf{p}^* \cdot \mathsf{q} \cdot \mathsf{r} + \mathsf{q} \quad \leq \quad \mathsf{p}^* \cdot \mathsf{q} \tag{8}$$

Consider (8). By unwindR, we know that

$$\mathsf{p}^* \cdot \mathsf{q} \quad = \quad (1 + \mathsf{p}^* \cdot \mathsf{p}) \cdot \mathsf{q}$$

By distrR, we know that

$$(1 + p^* \cdot p) \cdot q \quad = \quad 1 \cdot q + p^* \cdot p \cdot q$$

By commut+, we know that

$$1 \cdot q + p^* \cdot p \cdot q \quad = \quad p^* \cdot p \cdot q + 1 \cdot q$$

By id.L, we know that

$$p^* \cdot p \cdot q + 1 \cdot q \quad = \quad p^* \cdot p \cdot q + q$$

By mono+R, it suffices to show that

$$p^* \cdot q \cdot r \quad \leq \quad p^* \cdot p \cdot q \tag{9}$$

Consider (9). By mono.L, it suffices to show that

$$q \cdot r \quad \leq \quad p \cdot q \tag{10}$$

Consider (10). By =¡, it suffices to show that

$$q \cdot r \quad = \quad p \cdot q \tag{11}$$

Consider (11). By sym, it suffices to show that

$$p \cdot q \quad = \quad q \cdot r \tag{12}$$

Consider (12). By (1), we have what we need. □